I. To add no. having like signs [illegible] and prefix the common sign.

II. To add no. [illegible] unlike sign [illegible] of the [illegible] greater

III. [illegible]

[illegible]

Multiplication of [illegible]

IX. [illegible] 2 [illegible]

[illegible]

X. [illegible] 2 [illegible]

[illegible]

XI. When [illegible]

you [illegible] has a + [illegible]

[illegible]

[illegible] - sign

[illegible] 7, 1924 [illegible]

severe case of [illegible]

FIRST COURSE IN ALGEBRA

Bernice Manley

BY

W. H. WILLIAMS, M. A. (WILLIAMS COLLEGE)
HEAD OF THE DEPARTMENT OF MATHEMATICS, STATE NORMAL
SCHOOL, PLATTEVILLE, WISCONSIN

AND

W. B. KEMPTHORNE, PH.M. (UNIVERSITY OF WISCONSIN)
INSTRUCTOR IN MATHEMATICS, UNIVERSITY
OF OREGON, EUGENE, OREGON

Bernice
Opal
Manley

LYONS & CARNAHAN
CHICAGO NEW YORK

gk
1921
$1.00

[illegible]

[illegible]

N. H. S.

[illegible]

COPYRIGHT 1913, 1921

LYONS & CARNAHAN

[illegible] that these books

[illegible] of your life,

[illegible]

[illegible]

[illegible]

[illegible] Do

[illegible] of [illegible]

[illegible]

When you get old and [illegible]
[illegible]

PREFACE

This book is intended for use in the average high school. In its preparation, the authors have had in mind several features which they wished to emphasize:

I. To correlate the work closely with arithmetic. In addition to this close correlation, the fact is emphasized throughout that in handling algebraic expressions, the student is simply dealing with numbers.

II. To introduce but one new difficulty at a time. For example, the student is first made familiar with the new number symbols, and then his concept of number is enlarged to include negative numbers.

III. To emphasize the equation. At the very beginning of the book, the equation is used in solving a simple problem. As the student advances, new sets of equations which require new operations are introduced. The solution of equations by factoring is made prominent. In all, eight chapters are devoted specifically to the equation, besides which, equations for solution are given in nearly every other chapter.

IV. To lay stress on mental work. Repeated suggestions are made in the earlier part of the book that the exercises be solved mentally. It has not been thought necessary to make these suggestions so often in the later pages, as the student will not resort to the pencil if he has been trained in the earlier work to refrain from doing so when possible.

V. To give a large proportion of simple exercises, so that the principle involved shall not be lost sight of, as it often is in more complicated exercises.

VI. To emphasize algebraic language. To this end there are frequent calls for translation from algebraic language into English and from English into algebraic language. The review exercises offer an additional opportunity for such training.

The chapter on Quadratics has been placed before that on Exponents and Radicals, and the simplification of radicals needed for the solution of the quadratics has been given in connection with evolution. This order has been followed because, in many high schools, both subjects are not included in the first year course and it is believed that if one of these subjects must be omitted, it had better be radicals. If both can be taken, the authors advise taking the chapter on Exponents and Radicals first.

The review exercises given at the close of most of the chapters constitute an important feature of the book. They not only give a review of the immediate chapter, but bring up repeatedly the more important and more difficult parts of previous chapters.

The book has been used in several classes, while in manuscript form, and has been found workable. There are more exercises and problems given in some sections than should be demanded of most classes and the teacher should feel free to omit wherever it seems advisable. The stronger students should be encouraged to work the problems omitted in the regular class assignments.

The book contains about the same amount of material as the average algebra, but has from fifty to sixty more pages. This increase in size is partly due to the unusual number of exercises given, including many oral exercises, and partly to the additional space necessary for the arithmetical approaches to the various phases of the subject; but it is largely due to the desire to avoid the crowded pages that make the average algebra so unattractive, so difficult to read, and so disheartening to the student. Type of good readable size has been used and small-type paragraphs avoided. Page 124 is a fair sample of the open spacing between problems in exercise groups. It is believed that the attractiveness of the text and its value to the student have been vastly increased by the typographical style followed.

The authors acknowledge their indebtedness to a large number of teachers for suggestions and assistance of various kinds. They are particularly indebted to I. N. Warner of the State Normal School, Platteville, Wis., and to Alfred Livingston, Principal Lockport, Ill., high school.

CONTENTS

Chapter		Page
I.	INTRODUCTION	1
II.	LITERAL NUMBERS	18
III.	NEGATIVE NUMBERS	31
IV.	ADDITION	48
V.	AXIOMS AND EQUATIONS	59
VI.	SUBTRACTION	67
VII.	SYMBOLS OF GROUPING	73
VIII.	CONDITIONAL EQUATIONS AND IDENTITIES	78
IX.	MULTIPLICATION	85
X.	DIVISION	95
XI.	SIMPLE EQUATIONS—CONTINUED	106
XII.	SPECIAL PRODUCTS AND QUOTIENTS. FACTORS	113
XIII.	SOLUTION OF EQUATIONS BY FACTORING	148
XIV.	HIGHEST COMMON FACTOR. LOWEST COMMON MULTIPLE	155
XV.	FRACTIONS	162
XVI.	EQUATIONS CONTAINING FRACTIONS	194
XVII.	GRAPHS	205
XVIII.	SIMULTANEOUS EQUATIONS	220
XIX.	INVOLUTION AND EVOLUTION	247
XX.	QUADRATIC EQUATIONS	273
XXI.	SIMULTANEOUS QUADRATICS	296
XXII.	EXPONENTS AND RADICALS	305
XXIII.	RATIO, VARIATION, AND PROPORTION	333

Presented by Ramsey Har
J. M. H. M.H.S.
19-24 1920

FIRST COURSE IN ALGEBRA

CHAPTER I

INTRODUCTION

1. Algebra is one of the most important branches of mathematics. Like arithmetic, it deals with numbers. In arithmetic, numbers are usually represented by Arabic figures. In algebra, *letters* as well as Arabic figures are used. This use of letters often shortens and simplifies the solution of a problem, as the following illustrates:

In a certain school of three departments there are 90 pupils. There are three times as many pupils in the primary and twice as many in the intermediate as in the grammar department. How many pupils are there in each department?

Solution by arithmetic:

The number of pupils in the primary department equals three times the number of pupils in the grammar department.

The number of pupils in the intermediate department equals twice the number of pupils in the grammar department.

Hence, the number of pupils in the three departments equals six times the number of pupils in the grammar department, or 90.

Then the number of pupils in the grammar department equals $\frac{1}{6}$ of 90, or 15; the number in the intermediate department equals 2 times 15, or 30; and the number in the primary department equals 3 times 15, or 45.

The arithmetical solution given is based upon the number of pupils in the grammar department. The numbers of pupils in the other departments and in the school are expressed as multiples of the number in the grammar department. In algebra, for brevity, a symbol such as n or x replaces the long expression "the number of pupils in the grammar department."

Solution by algebra:

Let $n =$ the number of pupils in the grammar department.
Then $2n =$ the number of pupils in the intermediate department,
and $3n =$ the number of pupils in the primary department.
Therefore $6n =$ the number of pupils in all departments.
The problem tells us this number is 90.
Hence $6n = 90$,
$n = 15$, the number in the grammar department,
$2n = 30$, the number in the intermediate department,
and $3n = 45$, the number in the primary department.

PROBLEMS

Solve the following problems, using x or some other letter to represent the number, or one of the numbers, to be found:

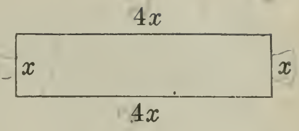
1. One of two numbers is twice the other and their sum is 150. Find the numbers.

2. Twice a number added to the number gives 42. What is the number?

3. A father's age is three times his son's age and the sum of their ages is 48. What are their ages?

4. Separate $100 into two parts, one of which is four times the other.

5. A rectangle is four times as long as it is wide and its perimeter (distance around) is 200 feet. Find its length and width.



6. A rectangle is twice as long as it is wide and its perimeter is 48 inches. What are its length and width?

7. A rectangle is three times as long as it is wide, and its perimeter is 96 feet. Find its length and width.

8. A boy buys a dozen oranges and a dozen bananas for 60 cents. The oranges cost twice as much as the bananas. What is the cost of the oranges?

9. A boy paid $2.25 for a ball and a bat. The ball cost twice as much as the bat. What was the cost of each?

10. A man paid $18 for railroad tickets for himself and son. His own ticket cost twice as much as his son's. What did each cost?

11. A boy has 12 coins consisting of quarters, dimes, and nickels. He has twice as many dimes as quarters, and three times as many nickels as quarters. How many coins of each kind has he?

12. During a certain month a merchant paid four times as much for salaries as for fuel, and half as much for rent as for salaries. He paid $175 for all. What was the amount paid for each?

13. The greater of two numbers is six times the smaller, and their difference is 65. Find the numbers.

14. The Arkansas river is four times as long as the Des Moines, and the sum of their lengths is 2500 miles. What is the length of each?

15. A lady paid twice as much for a hat and half as much for a pair of gloves as she paid for a pair of shoes. For all she paid $14. What did she pay for each?

16. A farm yielded twice as many bushels of oats as wheat and four times as many bushels of corn as oats. The total yield was 1683 bushels. What was the yield of each?

17. In the 63d Congress, California and Iowa had the same number of representatives, and Ohio had twice as many as Iowa. These three states together had 44 representatives. How many had each?

18. The sum of three numbers is 72. The second number is three times the first, and the third is four times the first. What are the numbers?

19. The sum of three numbers is 56. The second is three times the first and the third is equal to the sum of the other two. What are the numbers?

2. Literal Numbers. Arabic figures represent special, or definite, numbers. That is, any figure, as 5, or any arrangement of two or more figures, as 385, represents only one number. On the other hand, letters represent general numbers; that is, any letter, as a or x, may represent *any* number. In the problems just solved, x (or whatever letter was used) represented a different number in each of the different problems. Numbers represented by letters are sometimes called literal numbers.

3. General Numbers. To illustrate the use of letters for general numbers let us consider the following four problems. The first three use only definite numbers. In the fourth, letters are used to represent general numbers.

(1) Find the cost of 7 apples at 3 cents each.

(2) Find the cost of 160 acres of land at \$75 an acre.

(3) Find the cost of $8\frac{1}{2}$ yards of cloth at \$.20 a yard.

How do you find the cost in problem 1? What is the 7? the 3 cents? How do you find the cost in problem 2? What is the 160? the \$75? How do you find the cost in problem 3? What is the $8\frac{1}{2}$? the \$.20?

In each case the cost is found by multiplying the price of one thing by the number of things bought. More briefly,

$$\text{Cost} = \text{number of things bought times price of each.} \tag{a}$$

These problems are of the same class or kind and by the substitution of letters for figures may be stated as a single problem, thus:

(4) Find the cost of n articles the price of which is p each. Problems 4 and 1 are the same, provided n is 7 and p is 3 cents. Problems 4 and 2 are the same, provided n is 160 and p is \$75. Problems 4 and 3 are the same, provided n is $8\frac{1}{2}$ and p is \$.20. In each case, if c represents the cost, then

$$\mathbf{c = n \times p} \tag{b}$$

The same fact is expressed by (b) as by (a), but in a briefer, more compact form. The algebraic statement $c = n \times p$ is called a **formula** and by it any problem of this kind may be solved by replacing n and p by the numbers they represent. For example, in problem 3, n is $8\frac{1}{2}$ and p is \$.20. Therefore $c = 8\frac{1}{2} \times \$.20 = \$1.70$.

What numbers do c, n, and p represent in problem 1? in problem 2?

EXERCISES

1. Find the cost of 15 algebras at \$1.10 each. What numbers do c, n, and p represent in this problem?

2. Find the cost of 16 yards of cloth at 20 cents a yard. What numbers do c, n, and p represent in this problem?

3. Find the cost of 14 dozen eggs at 18 cents a dozen. What numbers do c, n, and p represent?

4. Find the cost and state what numbers c, n, and p represent in each of the following:

18 boxes of strawberries at $12\frac{1}{2}$ cents a box.
$7\frac{3}{4}$ yards of cloth at 75 cents a yard.
$17\frac{1}{2}$ bushels of oats at 48 cents a bushel.
25 pounds of sugar at $5\frac{1}{2}$ cents a pound.

To illustrate further the use of letters for general numbers, in problems of a different kind, consider the following:

What is the interest on \$300 for 2 years at 6%?
Solving this problem as in arithmetic, we have
.06 of \$300 = \$18, the interest for one year
$2 \times \$18$ = \$36, the interest for two years
What is the 300? the 2? the 6%?

Write a statement like (a) on page 5, showing how to find the interest in any problem when the principal, rate, and time are given.

If i represents the interest, p the principal, and r the rate (in hundredths), and t the time (in years), we have

$$\mathbf{i = p \times r \times t = t \times r \times p} \qquad \text{(c)}$$

Substituting for t, r, and p the values given in the above problem, we have

$$i = 2 \times .06 \times \$300 = \$36$$

EXERCISES

Use formula (c) to solve the following problems:

What is the interest on:

1. \$600 for 2 years at 5%?
2. \$400 for $2\frac{1}{2}$ years at 6%?
3. \$625 for 1 year 8 months at 6%?
4. \$14 for $3\frac{1}{2}$ years at 4%?
5. \$640 for 2 years 6 months at 7%?
6. \$1200 for 3 years 4 months at 5%?
7. \$124 for 1 year 10 months at 6%?

4. The **symbols of operation** used in arithmetic for addition (+), subtraction (−), multiplication (×), division (÷), and for square root ($\sqrt{\ }$), are also used in algebra. The symbol = means equals or equal.

A dot placed between the numbers is often used as a sign of multiplication. $a \cdot b$ means a times b. $5 \cdot 7$ means 5 times 7. Multiplication of literal numbers is commonly indicated by writing the letters together without any sign. Thus $abc = a \cdot b \cdot c = a \times b \times c$.

Division is commonly indicated by writing the numbers to be divided in the form of a fraction, the dividend as the numerator and the divisor as the denominator of the fraction. $\frac{a}{b}$ is read "a divided by b."

5. Factors. Numbers, which, multiplied together, form a product, are factors of that product. Thus 3 and 7 are factors of 21. 5, a, b, and x are factors of $5abx$. Any combination of the above factors, such as ab, $5a$, ax, $5ax$, is a factor of $5abx$. Any divisor of a number is a factor of the number. What are the factors of $7cmx$?

6. Coefficient. Any factor of a number is the coefficient of the product of the remaining factors. In $7abxy$, 7 is the coefficient of $abxy$, $7a$ is the coefficient of bxy, $7ab$ is the coefficient of xy, $7ay$ is the coefficient of bx, xy is the coefficient of $7ab$, etc. The coefficient tells how many times the rest of the product is taken as an addend.

If the coefficient is a definite number, it is called the numerical coefficient and is usually written first. 7 is the numerical coefficient in $7abxy$. The coefficient 1 is usually omitted. The numerical coefficient in cx is 1, as though the term were written $1cx$.

What is the coefficient of ck in $5ck$? of k? of c? of $5k$?

What is the coefficient of x in $6axy$? of a? of ay?

What is the coefficient of cd in $3acds$? of ds? of ads?

7. Exponent. An exponent is a number written to the right and above another number to indicate how many times the latter is used as a factor.

(This definition applies to positive integral exponents only. Fractional and negative exponents will be defined later.)

$a \times a$ is usually written a^2, and is read "a square."
$a \times a \times a$ is usually written a^3, and is read "a cube."
$a \times a \times a \times a \times a$ is usually written a^5, and is read "a exponent 5."

In a^5, 5 is called the exponent of a. The exponent 1 is usually omitted. In $7x$ the exponent of x is 1, as though the term were written $7x^1$.

What is the value of 5^2? Ans. $5^2 = 5 \times 5 = 25$.

Find the values of the following: 5^3, 3^3, 5^4, 2^5, 7^2, 6^3, 4^6, 2^7, 15^2, 8^3, 14^2.

8. Power. A power of a number is the product obtained by using that number one or more times as a factor. a^2 is called "the second power of a" or "a square"; a^3 is called "the third power of a", or "a cube"; a^5 is called "the fifth power of a", or "a exponent 5."

9. Symbols of Grouping. A parenthesis is used when it is desired to indicate that the expression within is to be regarded as a single number. $10 - (5 + 3)$ means that the number, $5 + 3$, is to be subtracted from 10. Hence, $10 - (5 + 3) = 10 - 8 = 2$. $5(3 + 7)$ means 5 times the number, $3 + 7$, that is, $5 \times 10 = 50$.

Other symbols of grouping are the braces $\{\ \}$, brackets $[\ ]$, vinculum $\overline{\qquad}$, and fractional bar, as in $\frac{3}{4}$. In algebra, for convenience, these are sometimes included under the term parentheses.

What does $7 + 2(2 + 3)$ equal? $24 - (3 + 5) = ?$

10. An **algebraic expression** is an algebraic symbol or a combination of algebraic symbols that represents a number. Thus, a, $5a^2$, $3a^2 - 2bc$, and $3a^2 + 2bc$, are all algebraic expressions.

A **term** is an algebraic expression that does not contain a plus or minus sign. Thus, the expression $5a^2$ is a term.

Since an expression within a symbol of grouping is to be regarded as a single number, $3(a + b)$, $7(x - y)$, and $\sqrt{2a - x}$ are terms, even though, within the parenthesis, the plus or minus sign is found.

c^2, $5a^2x^3$, and $(4x + 2z)$ are terms. In the expression $a^3 + 5ax - (3x^2+5)$, the terms are a^3, $5ax$, and $(3x^2 + 5)$.

11. Interpretation of Algebraic Expressions. If a represents *any* number and b represents *any* number, what does $a + b$ represent?

EXERCISES

State in words (orally) what each of the following expressions represents:

1. $5c$ = 5·c
2. a^2 4
3. a^3 27
4. $a - x$ [illegible]
5. $2(x + y)$ = 2x+2y
6. $c + 3$ = 6
7. $a^2 + x$ 5
8. $x^2 + y^2$ x [illegible]
9. $\sqrt{a^3}$ 9
10. $(a + b)^2$ = 6
11. $c + 2d$ = 7c
12. $3a^2$ / [illegible]
13. $2x^2 - y$
14. $a + b + c$ = 5c
15. $m^2 - n^2$
16. $a^3 + y^3$
17. $(a + b + c)^2$
18. $2(a + b + c)^2$ 4
19. $3c - 2m$
20. $\sqrt{x + y}$
21. $(m + k)^3$
22. $5x - 3a$ = c
23. $8(z - 2y)$
24. $2y + x^2$

12. Value of an Algebraic Expression. The value of an algebraic expression is the number it represents when each letter is given a definite value. To reduce this value to its simplest form, perform all operations as indicated.

In finding the value of an algebraic expression, attention should be given to the following rules concerning the *order* of the operations:

1. In a term containing a symbol of grouping, operations inside the symbol of grouping are usually performed before those outside. Thus, $5(3 + 4) = 5 \cdot 7 = 35$.

2. In any term, operations involving raising to a power or extracting a root must be performed before multiplications or divisions, unless the latter are within a symbol of grouping.

Thus, $3 \cdot 2^3 = 3 \cdot 8 = 24$. But $(3 \cdot 2)^3 = 6^3 = 216$.

3. Multiplications and divisions are to be performed, in their order from left to right, before the term that contains them is combined with other terms.

Thus, $5 + 3 \cdot 4 - 5 + 18 \cdot 2 \div 6 = 5 + 12 - 5 + 6 = 18$

2 x

ILLUSTRATIVE EXERCISES

Find the value of $2a + 3k^2 - ak$ if $a = 5$ and $k = 2$.

Solution: $2a = 2 \cdot 5 = 10$

$3k^2 = 3 \cdot 2^2 = 3 \cdot 4 = 12$

$ak = 5 \cdot 2 = 10$

Hence, $2a + 3k^2 - ak = 10 + 12 - 10 = 12$

Find the value of $a^2 + 2ab + 3bc$ if $a = 1$, $b = 2$, and $c = 3$.

Solution:

$a^2 = 1^2 = 1$; $2ab = 2 \cdot 1 \cdot 2 = 4$; $3bc = 3 \cdot 2 \cdot 3 = 18$.

Hence, $a^2 + 2ab + 3bc = 1 + 4 + 18 = 23$.

EXERCISES

If $a = 2$, $b = 1$, $c = 3$, and $d = 4$, find the value of each of the following expressions:

1. a^2
2. $3a$
3. $4a^2$
4. $3b^3$
5. $a + b$
6. $3a - b$
7. $2a - 3b + 2c - d$
8. $6c - 2abc + 2d^2$
9. $5a + 3c^2 + d^3$
10. $2d - 3b + 3ad$
11. $(a + b)^2$
12. $\frac{d}{a} + 7c - 2a^2$
13. $(ad - 3b)^2$
14. $\frac{6c}{ab} - \frac{cd}{a}$
15. $12ac - (c + d)^2$

16. What is the value of $a^2 + 3a + 5$ when $a = 1$? when $a = 2$? when $a = 3$? when $a = 4$?

17. What is the value of $x^2 + 7x - 18$ when $x = 2$? when $x = 3$? when $x = 5$? when $x = 7$?

18. What is the value of $x^3 + 3x^2 + 10x$ when $x = 1$? when $x = 2$? when $x = 3$? when $x = 4$?

The process of finding the value of an algebraic expression is called **evaluation.**

If $a = 1$, $b = 10$, $m = 2$, $p = \frac{1}{2}$, $r = 9$, and $x = 10$, evaluate each of the following expressions:

1. $b - 3am + x^2$
2. $4px^2 - r + mx$
3. $8bp^3 + 3a^2m^2x - rx$
4. $b^2p + 6m^3p^2 - \sqrt{abx}$
5. $16x^3 - 3rp^2 + \sqrt{5bm} - 4$
6. $\frac{bm}{x} + \frac{5b}{m} - \frac{px}{a}$
7. $\frac{brx}{2m} - 4pr + x^2$
8. $b^m + 2rx$
9. $mx - 2apr$
10. $2aprx - m$
11. $3br + r^mx$
12. $3(a + b) - 2r$

13. Substitution in Formulas. An important application of algebra is found in the use of formulas. Students of geometry, physics, chemistry, and other branches of learning must frequently substitute values for letters in formulas which they wish to use.

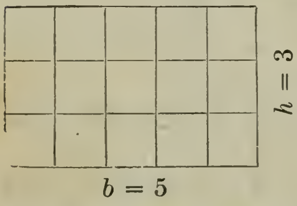
Up to this point two formulas have been used: *viz.*,

$\mathbf{c = n \cdot p}$ (*i.e.*, cost = number × price), and

$\mathbf{i = t \cdot r \cdot p}$ (*i.e.*, interest = time × rate × principal).

Another formula frequently used is that for the area of a rectangle. The area of a rectangle is equal to the product of its base and altitude. If we represent the number of units in the area by A, the number of linear units in the altitude (or *height*) by h, and in the base by b, the formula for the area is: $\mathbf{A = bh}$

ILLUSTRATIVE EXERCISE



What is the area of a rectangle the base of which is 5 inches and the altitude of which is 3 inches?

Solution:

Here $b = 5$ and $h = 3$. Hence, $A = 5 \cdot 3 = 15$, *i.e.*, the area is 15 square inches.

EXERCISES

Find the area of each of the following rectangles:

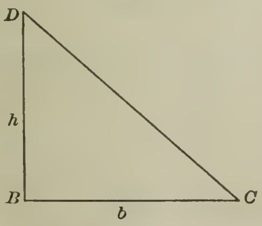
1. $b = 10$ in., $h = 6$ in.
2. $b = 6$ ft., $h = 3$ ft.
3. $b = 18$ in., $h = 10$ in.
4. $b = 8$ ft., $h = 2.4$ ft.
5. $b = 20$ rods, $h = 20$ rods
6. $b = 100$ rods, $h = 40$ rods
7. $b = 120$ cm. (centimeters), $h = 80$ cm.
8. $b = 15$ yd., $h = 10$ yd. 2 ft.

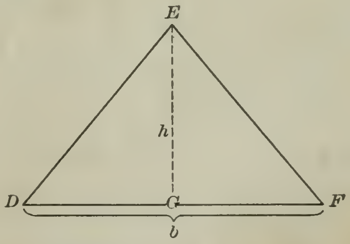
9. How many acres are there in a rectangular field the length of which is 50 rods and the width of which is 32 rods?

10. How many acres are there in a rectangular field which is 40 rods long and 30 rods wide?

11. How many acres are there in a rectangular field the length of which is 60 rods and the width 30 rods?

The area (A) of a triangle is one-half the product of the base (b) and altitude (h).





State in algebraic language the formula for the area of a triangle.

ILLUSTRATIVE EXERCISE

1. What is the area of a triangle the base of which is 4 feet and the altitude 3 feet?

Solution:

Here $b = 4$ and $h = 3$. Substituting these values for b and h in the formula, we have

$$A = \frac{3 \cdot 4}{2} = \frac{12}{2} = 6.$$

Find the area of a triangle if

2. $b = 5$ in., $h = 12$ in.
3. $b = 14$ yd., $h = 20$ yd.
4. $b = 60$ cm., $h = 25$ cm.
5. $b = 24$ ft., $h = 50$ ft.
6. $b = 35$ rods, $h = 40$ rods.
7. $b = 2.5$ ft., $h = 16$ ft.

The formula for the area of a circle in terms of its radius, r, is

$$\mathbf{A = \pi r^2}$$

π (pronounced *pī*) is an arbitrary symbol the value of which is approximately $\frac{22}{7}$. Exactly, its value is equal to the circumference of a circle divided by its diameter.

ILLUSTRATIVE EXERCISE

1. Find the area of a circle if $r = 6$ inches.

Solution:

Substituting the values of π and r in the formula $A = \pi r^2$, we have

$$A = \frac{22}{7} \times 6^2 = \frac{22}{7} \times 36 = 113\tfrac{1}{7}.$$

The area of the circle is $113\frac{1}{7}$ square inches.

EXERCISES

Find the area of a circle whose radius is

1. 5 inches
2. 7 inches
3. 8 feet
4. 17 cm.
5. 12 rods
6. 15 meters

7. How many square yards are there in a circular piece of ground the radius of which is 7 feet?

8. How many acres are there in a circular race track the radius of which is 51 rods?

14. Historical Note. The use of symbols to denote general numbers dates back to the earliest mathematical manuscript, that of an Egyptian priest, Ahmes (about 1700 B.C.). This manuscript is supposed to be based on a much older work. Ahmes called the number to be found "Hau," meaning "heap." He had several problems of the following character: "Heap, its seventh, its whole, it makes 19," which means "Find a number which added to one-seventh of itself equals 19." Ahmes gave the correct answer: *viz.*, $16\frac{5}{8}$.

The Egyptians advanced little beyond this work of Ahmes. The Greeks devoted their attention chiefly to geometry, but made some progress in algebra. Aristotle (350 B.C.) represented numbers by letters, and by the time of Pappus (280 A.D.) capital letters were in use for that purpose. A little later Diophantus of Alexandria (about 325 A.D.) used abbreviations for the numbers and operations in constant use. The Hindoos also made considerable progress in this direction, but the credit for the symbolic algebra we have today is largely due to Vieta, a Frenchman, (1550–1603), and to Harriot, an Englishman, (1560–1621). While others had used letters to represent numbers, these made them an essential part of algebra.

REVIEW EXERCISES

15. 1. How are general numbers represented? How do they differ from arithmetical numbers?

2. If $a = 5$, give the values of the following: $2a$, a^2, a^3, $a^2 + 2a - 3$.

Beginning [illegible] find the [illegible]

3. Evaluate $x^2 - 2x + 7$, when $x = 2$; when $x = 3$.

4. The sum of three numbers is 18. The second is twice the first and the third is equal to the sum of the other two. What are the numbers?

5. John has five times as many marbles as his brother, and together they have 42 marbles. How many has each?

6. Express the following in words: $x + b - c$, $2a + 2$, $a + 4$, $a + b - 7$.

7. Express algebraically a number one more than a.

8. Express algebraically the sum of a and b; the sum of a and 10.

9. Express algebraically the product of x and y; the quotient of x divided by y.

10. Define *factor*, *coefficient*, *exponent*, and *power*, and give an example of each.

11. Evaluate $a^3 - 2a^2 + 6a + 3(a + 2)$, if $a = 3$.

12. A boy paid two dollars for a hat and a knife. The hat cost three times as much as the knife. What did he pay for the hat?

13. A baseball team played 120 games. The number of games won was twice the number of games lost. How many games did the team win?

14. If $a = 2$, $b = 7$, $c = 5$, $d = 1$, and $e = 3$, evaluate each of the following expressions:

(a) $\dfrac{3bc - a^2c - e^3}{b - e - a}$

(b) $b\sqrt{2ac^2} - 7ad - b^2d$

15. Who wrote the oldest mathematical work now known?

16. Write a number a less than x^2. Write a number 4 less than the product of a and x.

17. Find the areas of the following:

(a) A rectangle in which $b = 18$ in., $h = 10$ in.

(b) A rectangle in which $b = 22$ ft., $h = 6$ ft.

18. Express the following in words:

$$(x + \underset{49}{y})^2 = x^2 + 2xy + y^2.$$

$$(x + \underset{21}{y})(x - y) = x^2 - y^2.$$

19. Show that the statements in problem 18 are true if $x = 5$, and $y = 2$.

SUMMARY

SECTION
1. Relation between algebra and arithmetic. Algebraic solution of problems.
2. Literal numbers.
3. General numbers.
4. Symbols used in algebra.
5. Factor. Definition, examples.
6. Coefficient. Definition, examples. Numerical coefficient.
7. Exponent. Definition, examples.
8. Power. Definition, examples.
9. Symbols of grouping. Definition, examples.
10. Algebraic expression. Term. Definitions, examples.
11. Interpretation of algebraic expressions. Examples.
12. Evaluation of algebraic expressions.
13. Substitution in formulas. Problems.
14. Historical note.
15. Review exercises.

CHAPTER II

LITERAL NUMBERS

16. Problems Leading to Literal Expressions.

1. How many pints are there in one quart? in x quarts?

2. How many ones are there in one ten? in x tens?

3. How many cents are there in n dimes? in y dollars?

4. How many minutes are there in 5 hours? in a hours?

5. How many quarts are there in x pecks? How many inches are there in c feet?

~~6. How~~ many square rods are there in c acres?

7. What do 7 yards of cloth cost at x cents a yard?

8. What is the cost of b cans of pears at 15 cents a can?

9. What is the cost of a yards of cloth at k cents a yard?

10. If the value of x is 10, what is the value of $x + 4$? $x - 7$? $7x$? $\frac{x}{3}$?

11. Write a number which is 3 more than a; 3 less than a; 3 times a; one-third of a.

12. Write a number which is b more than a; b less than a.

13. Write a number which is 6 greater than x; 7 less than 2 times b; 2 more than 4 times b; greater than y by 4.

30

[illegible]

14. What are consecutive numbers? If 6 is the smaller of two consecutive numbers, what is the larger? If x represents the smaller of two consecutive numbers what is the larger?

15. What number is 5 more than the square of x?

16. How do you express algebraically the difference between the square of a and the square of b if a is greater than b?

17. A and B together have \$20. A has x dollars; how many has B?

18. A piece of cloth contains x yards. If y yards are cut off, how many yards remain?

19. What is the interest on p dollars for 1 year at 5%? 2 years? m years? 1 month? 4 months? x months? c years and d months?

To here.

20. If a man can walk a certain distance in 4 hours, what part of the distance can he walk in 1 hour? 3 hours? b hours? $b + 1$ hours?

21. If a man can walk a certain distance in x hours, what part of the distance can he walk in 1 hour? 3 hours? m hours?

22. If A can do a piece of work in 12 days, what part of the work can he do in 1 day? in 7 days? in h days? in $m + n$ days?

23. If A can do a piece of work in y days, what part of the work can he do in 1 day? in 3 days? in p days?

24. A's age is now 30 years. What will his age be in 4 years? in m years? What was his age 5 years ago? b years ago?

25. The sum of two numbers is 24 and one of them is n. What is the other? The sum of two numbers is s and one of them is x. What is the other?

$d + x$

26. If the difference between two numbers is d and the smaller is x, what is the greater?

27. The product of two numbers is 35 and one of them is 7. What is the other? If one of the numbers is x, what is the other?

28. If the quotient of two numbers is 2 and the divisor is 5, what is the dividend? If the divisor is c, what is the dividend?

29. A newsboy received x cents for 12 papers. What did he receive for each paper?

30. What is the perimeter of a rectangular lot x feet long and 60 feet wide?

31. What will be the total cost of b yards of cloth at 15 cents a yard and c yards at 25 cents a yard?

32. A rectangular field is x rods long and y rods wide. What is the area of the field in square rods? in acres? What is the value of the field at m dollars an acre?

3, 4 - 7 M

17. Addition. When a factor occurs in each of two or more numbers, it is called. a common factor of these numbers. Thus 2 is a common factor of 6, 10, and 14; a is a common factor of $3a$, $2ax$, and $5ab$.

In solving the problems in section 1, numbers having a common factor, such as $5x$, $3x$, and x, were added. Further addition problems of the same character are here given.

What is the sum of 5 quarts and 3 quarts? of 3 miles and 4 miles?

A man walks 6 miles one day and 5 miles the next day. How far does he walk in the two days?

There are 8 quarts of milk in one pail and 6 quarts in another. How many quarts of milk are there in the two pails?

5 yards, 4 yards, and 2 yards are how many yards?

$5 \cdot 2$ and $3 \cdot 2$ are how many twos?

$6 \cdot 3$, $4 \cdot 3$, and $2 \cdot 3$ are how many threes?

$5a$, $4a$, and $10a$ are how many a?

These exercises suggest the following rule:

To add numbers having a common factor:—

Write the common factor and as its coefficient the sum of its coefficients in the given numbers.

ILLUSTRATIVE EXERCISES

What is the sum of $3a$, $6a$, and $7a$?

It is convenient to write the numbers in a column as follows:

$$\begin{array}{r} 3a \\ 6a \\ 7a \\ \hline 16a \end{array}$$

The common factor a is brought down and 3, 6, and 7, the coefficients of a, are added and written as the coefficient of a in the sum, giving $16a$.

What is the sum of $3 \cdot 2 \cdot 7$, $5 \cdot 2 \cdot 7$ and $7 \cdot 2 \cdot 7$?

$$\begin{array}{r} 3 \cdot 2 \cdot 7 \\ 5 \cdot 2 \cdot 7 \\ 7 \cdot 2 \cdot 7 \\ \hline 15 \cdot 2 \cdot 7 \end{array}$$

The common factor is $2 \cdot 7$. We write this factor and, as its coefficient, the sum of its coefficients in the given numbers, as follows: $3 + 5 + 7 = 15$. The sum is $15 \cdot 2 \cdot 7$. Check this result by adding in the usual way.

$$\begin{array}{r@{\;=\;}r} 3 \cdot 2 \cdot 7 & 42 \\ 5 \cdot 2 \cdot 7 & 70 \\ 7 \cdot 2 \cdot 7 & 98 \\ \hline 15 \cdot 2 \cdot 7 & 210 \end{array}$$

What is the sum of $5(a + b)$, $6(a + b)$, and $4(a + b)$?

$$\begin{array}{r} 5(a + b) \\ 6(a + b) \\ 4(a + b) \\ \hline 15(a + b) \end{array}$$

The common factor is $(a + b)$ and its coefficients are 5, 6, and 4. Hence the sum is $15(a + b)$.

EXERCISES

Solve mentally the following problems:

1. 5 quarts + 6 quarts = ——— quarts.
2. $5 \cdot 4 + 6 \cdot 4 + 3 \cdot 4 =$ ——— 4.
3. $4c + 7c + 8c + 3c = ?$
4. $7c + 3c = ?$
5. $7 \cdot 4 + 3 \cdot 4 = ?$
6. $7t + 6t + 8t = ?$
7. $3 \cdot 9 + 4 \cdot 9 + 8 \cdot 9 = ?$
8. $7 \cdot 8 + 3 \cdot 8 + 6 \cdot 8 = ?$
9. $5s + 6s + 7s = ?$
10. $3k + 7k + 8k = ?$
11. $3 \cdot 7 + 4 \cdot 7 + 8 \cdot 7 = ?$
12. $5 \cdot 3 + 6 \cdot 3 + 4 \cdot 3 = ?$
13. $4m + 8m + 7m + 3m = ?$
14. $5k + 6k + 7k + 10k = ?$
15. $.3y + .4y + .7y = ?$
16. $.2a + .5a + 1.4a = ?$
17. $3x + 1.6x + .4x = ?$
18. $2(a + b) + 3(a + b) = ?$

19. $5(x + 2y) + 2(x + 2y) = ?$

20. $7(a + 6) + 3(a + 6) + 2(a + 6) = ?$

Perform the following additions, mentally when possible:

21. $4 \cdot 5 \cdot 8 + 6 \cdot 5 \cdot 8$ 10·5·8

22. $3 \cdot 4 \cdot 9 + 2 \cdot 4 \cdot 9 + 6 \cdot 4 \cdot 9$ 11·4

23. $4 \cdot 6 \cdot 3 + 4 \cdot 6 \cdot 5$ 8·4·6

24. $1.2m + .5m + 4.3m + 3m$ 9

25. $13.3c + 2.5c + .7c + 8.5c$ 25 c

26. $\frac{1}{2}d + \frac{3}{4}d + \frac{1}{4}d$ 1½ d

27. $\frac{2}{3}xy + \frac{5}{6}xy + \frac{1}{2}xy$ 2 xy

28. $\frac{3}{2}mn + \frac{1}{3}mn + \frac{3}{5}mn$ ~~$\frac{7}{30}$ mn~~ $2\frac{13}{30}$ mn

29. $3x^2 + 4x^2 + 7x^2 + 5x^2$ $19x^2$

30. $6a^2bc + .5a^2bc + 2.3a^2bc + .4a^2bc$ 9.2 a^2bc

31. $\frac{1}{2}x^3y + \frac{2}{5}x^3y + \frac{3}{10}x^3y + 3x^3y$ $4\frac{1}{5}x^3y$

18. Subtraction. Subtract 3 quarts from 5 quarts. Subtract $3k$ from $5k$.

ILLUSTRATIVE EXERCISE

Subtract $3 \cdot 4$ from $5 \cdot 4$.

Solution:

$$\begin{array}{r} 5 \cdot 4 \\ 3 \cdot 4 \\ \hline 2 \cdot 4 \end{array}$$

14

Check:

$$\begin{array}{r} 5 \cdot 4 = 20 \\ 3 \cdot 4 = 12 \\ \hline 2 \cdot 4 = 8 \end{array}$$

To subtract when the numbers have a common factor:

Write the common factor, and as its coefficient write its coefficient in the minuend diminished by its coefficient in the subtrahend.

EXERCISES

1. Take $3 \cdot 5$ from $7 \cdot 5$. Check the result.

2. Subtract $7 \cdot 8$ from $13 \cdot 8$. Check the result.

3. Subtract $9 \cdot 3$ from $16 \cdot 3$. Check the result.

4. Subtract $7k$ from $12k$

5. Subtract $8abx$ from $20abx$

6. Take $20\,c^2y$ from $32c^2y$

7. Take $9ar^2$ from $22ar^2$

8. From $42x$ take $15x$

9. From $54ck$ take $39ck$

10. From $13c^2y^3$ take $8c^2y^3$

11. From $54x$ take $22x$

12. $8 \cdot 3 \cdot 7 - 5 \cdot 3 \cdot 7 = ?$

13. $2 \cdot 7 \cdot 5 - 2 \cdot 3 \cdot 5 = ?$

14. From $34a^2x$ take $21a^2x$

15. From $18abx^2$ take $11abx^2$

16. $44x^2y^3 - 17x^2y^3 = ?$

17. $1.3mn - .9mn = ?$

18. $13.4k - 7.6k = ?$

19. $8(a - x) - 5(a - x) = ?$

20. $17(a^2 - b^2) - 9(a^2 - b^2) = ?$

21. $\begin{array}{r} 27x^2y \\ \underline{11x^2y} \\ 16x^2y \end{array}$

22. $\begin{array}{r} 15m^2x^2 \\ \underline{9m^2x^2} \\ 6m^2x^2 \end{array}$

23. $\begin{array}{r} 52ab^2d \\ \underline{34ab^2d} \\ 18ab^2d \end{array}$

24. $\begin{array}{r} .35uvx \\ \underline{.19uvx} \\ 16uvx \end{array}$

25. $\begin{array}{r} \frac{2}{3}mx \\ \underline{\frac{1}{3}mx} \\ 1mx \end{array}$

26. $\begin{array}{r} \frac{7}{8}bc^2 \\ \underline{\frac{3}{4}bc^2} \\ \frac{1}{8}bc^2 \end{array}$

27. $\begin{array}{r} 2.5d^2e^2 \\ \underline{1.7d^2e^2} \\ 3d^2e^2 \end{array}$

28. $\begin{array}{r} \frac{5}{2}bch^2 \\ \underline{\frac{2}{5}bch^2} \\ 2\frac{1}{10}bch^2 \end{array}$

19. Multiplication. Multiply 5 by 3. What is 3 times 5 quarts? 3 times $5x$? 3 times 5 twos?

ILLUSTRATIVE EXERCISE

Multiply $7 \cdot 3$ by 2

Solutions:

This multiplication may be carried out in one of the following ways:

$$\begin{array}{rl} & 2 \cdot 7 \cdot 3 = 7 \cdot 6 = 42 \\ \text{or,} & 2 \cdot 7 \cdot 3 = 14 \cdot 3 = 42 \\ \text{or,} & 2 \cdot 7 \cdot 3 = 2 \cdot 21 = 42 \end{array}$$

To multiply a product by a number:

Multiply one of its factors by that number.

EXERCISES

Perform (mentally) the following multiplications:

1. $3 \cdot 4 \cdot 5$
2. $3a \cdot 5$
3. $6ab \cdot 4$
4. $3as \cdot 6$
5. $7 \cdot 5 \cdot 4$ in three ways
6. $9ak \cdot 8$
7. $24abc \cdot 3$
8. $3bc \cdot a\ (= 3abc)$
9. $2bsx \cdot y$
10. $3ack \cdot p$
11. $5b^2x \cdot m$
12. $2a^2s \cdot x$
13. $14a \cdot 3x$ 72
14. $3 \cdot 7 \cdot 6$ in three ways
15. $3ab \cdot 7c^2$
16. $12a^2 \cdot 2.5b$ 120 a²b
17. $7xy \cdot .4$
18. $3kz \cdot 1.5$
19. $14a^2 \cdot 2b$
20. $22x^2 \cdot 3y$ = 66
21. $3.5mn \cdot 2$
22. $25b^2cx \cdot .5$
23. $15cx^2y^2 \cdot 4$ 600x
24. $\frac{2}{3}bc \cdot 6$
25. $\frac{4}{5}r^2s^2 \cdot 20$ 16
26. $\frac{1}{2}xyz^2 \cdot \frac{3}{4}$

20. Division. Divide 54 quarts by 3. Divide $54x$ by 3.

ILLUSTRATIVE EXERCISE

Divide $54 \cdot 6$ by 3.

Solutions:

$$\frac{54 \cdot 6}{3} = \frac{54}{3} \cdot 6 = 18 \cdot 6 = 108$$

$$\frac{54 \cdot 6}{3} = 54 \cdot \frac{6}{3} = 54 \cdot 2 = 108$$

$$\frac{54 \cdot 6}{3} = \frac{324}{3} = 108$$

To divide a product by a number:

Divide one of its factors by that number.

EXERCISES

Perform mentally the following operations:

1. $42 \div 6$ 9
2. $42 \cdot 12 \div 6$
3. $3abc \div b$ 3ac
4. $3abc \div c$ 3ab
5. $12abs \div 2a$ 6bs
6. $12abs \div 3s$
7. $24c^2k \div c^2$
8. $24c^2k \div 6$
9. $32km \div 8m$
10. $32km \div 4k$
11. $16n^2y \div 2y$
12. $1.2xy^2 \div 4$
13. $4x^2y^3 \div .5$
14. $64x^3z^2 \div 16x^3$
15. $7.5mn^2 \div 5n^2$ $1\frac{3}{2}m$
16. $3.2abc \div 4a$ 8bc
17. $\frac{72mn}{mn} = ?$ 72
18. $12(a + b) \div 3$ 4(a+B)
19. $27(a + b) \div (a + b)$ 27.
20. $\frac{4ax(m - n)}{2ax} = ?$ 2(m-n
21. $\frac{150x^2(r - s)}{15(r - s)} = ?$ $10x^2$
22. $2.5s(c + d) \div (c + d)$ 2.5
23. $\frac{3}{4}b^2c \div 3c$ [illegible]
24. $25ax(a + x) \div 5ax$ 5(a+x
25. $7.56x \div 3x$ 2.15

21. Signs of the Fundamental Operations. The Hindoos indicated addition by writing the numbers together without sign. This is still done in expressing the sum of an integer and a fraction. Thus $5 + \frac{2}{3}$ is written $5\frac{2}{3}$. Among other symbols that have been used is *p*, the initial letter of plus. Cardan (1501–1576) wrote $x^3 + 6x = 20$ as follows: *Cubus p 6 rebus aequalis 20.* The present sign for addition first indicated an excess, a use it still has, but by 1630 it was recognized as a symbol of operation.

Various symbols were used to denote subtraction, among them the dot (Hindoos), now a sign of multiplication, and the present signs of equality and division. The symbol now used was originally employed by merchants and others to denote a shortage. By 1630, it, like "+", had become a recognized symbol of operation.

The three ways of indicating multiplication found in this book came into use at about the same time. In 1631 Oughtred (1574–1660) introduced the symbol × and Harriot (1560–1621) employed the dot. Descartes also made use of the dot and in 1637 wrote the factors side by side, as *ab*.

The Arabs denoted division in the way now common in algebra; *viz.*, by writing the numbers in the form of a fraction, $\frac{329}{8}$. The dot, the colon, and the present sign for subtraction appear among the symbols for division. The present sign dates back to about 1660.

REVIEW EXERCISES

22. 1. State the rule for adding numbers having a common factor.

2. Add $3 \cdot 7$, $2 \cdot 7$, and $10 \cdot 7$. 15·7 = 105

3. $4am + 3am + 15am = ?$ 22am

4. State the rule for subtracting numbers having a common factor.

5. From $12 \cdot 9$ subtract $7 \cdot 9$. 5·9 = 45

6. $(6a^2 + 3a^2 + 7a^2) - 13a^2 = ?$

7. Write a number which is $2a$ more than 3 times b.

8. Write in algebraic symbols: Fourteen times the square of the sum of a and b minus the product of the sum of a and b by the difference of a and b. Evaluate the foregoing if $a = 3$ and $b = 2$.

7– 12 – 21 –

9. How is a product multiplied by a number?

10. Multiply 2·3·5 by 6 in three ways.

11. Find the product of 2, 5, and 7.

12. Find the product of $2a$, $3b$, and $4x$.

13. Multiply $12x^2y$ by $\frac{3}{4}$.

14. How is a product divided by a number?

15. Divide 4·6·10 by 2 in three ways.

16. $35ac + 7ac = ?$ $\quad \dfrac{3.2x^2mn}{8x^2} = ?$ $\frac{3mn}{4}$

42 ac

17. From the sum of $14m^2n$ and $11m^2n$ subtract the product of $3m^2$ and $4n$. 13 m2n.

18. Multiply the sum of $6axy$ and $12axy$ by $3b$. 54

19. The sum of three numbers is 49. The second is twice the first and the third is twice the second. What are the numbers? 7 — 14 - 28

20. What number added to $14ab^2$ gives $25ab^2$? 11

21. How far does a train go in a hours at x miles an hour?

22. What number subtracted from $19ax$ gives $3ax$? 16 x

23. What number multiplied by $2cd$ gives $24cdx$? 12x

24. A rectangular field is 3 times as long as wide and the distance around it is 160 rods. Find the length and width of the field.

25. If k is the middle one of three consecutive integers, what are the others? K + 1 k K - 1

26. What number divided by $5bx$ gives $4m$? 20[illegible]

27. Write a number 5 greater than twice the product of a and x.

28. Manager Jennings took 20 men to Cleveland for a series of ball games. In all, x men took part in the games. How many men did not play? 20 − x

29. How long will it take a man to walk c miles if he walks 3 miles an hour?

30. What is 5 per cent of 300? What is x per cent of 300?

In the number 64, 6 and 4 are called digits. 6 is the tens' digit and 4 the ones' digit. In terms of its digits $64 = 10 \times 6 + 4$.

31. If the ones' digit of a number is 3 more than the tens' digit and the tens' digit is x, what is the number?

32. If the ones' digit is x and the tens' digit is 2 more than the ones' digit, what is the number?

33. Express in words: $(x + 2a)(x - 2a) = x^2 - 4a^2$. Show that this is true if $x = 7$ and $a = 3$; if $x = 13$ and $a = 6$.

34. Express algebraically the sum of x ~~and~~ y; the difference between x ~~and~~ y.

35. If A and B are 25 miles apart and walk toward each other, A at the rate of 3 miles an hour and B at the rate of 2 miles an hour, how fast do they approach each other? In how many hours will they meet?

36. If A and B walk in the same direction with A behind, how much does A gain on B each hour? In how many hours will he overtake B? 2

37. If, in problem 35, A and B start m miles apart, in how many hours will they meet?

38. If, in problem 36, A and B are m miles apart, in how many hours will A overtake B?

39. A man sold a horse for \$250, thereby gaining x dollars. What did the horse cost?

40. Express k feet and m inches in inches.

SUMMARY

SECTION

16. Problems leading to literal expressions.
17. Addition of numbers having a common factor. Examples.
18. Subtraction when numbers have a common factor. Examples.
19. Multiplication. Examples.
20. Division. Examples.
21. Signs of the Fundamental Operations.
22. Review exercises.

$6x^2 = 12$ [illegible]

$a = 7$ [illegible]

CHAPTER III

NEGATIVE NUMBERS

23. Extension of Numbers. The number field of algebra is more extended than that of arithmetic. At first only whole numbers were used, but the necessity of expressing parts soon arose and fractions came into use. These were handled at first with great difficulty, and for a long time only unit fractions (that is, fractions with the numerator 1) or fractions having a common denominator were used in computation. Still later there came the need for expressing values such as $\sqrt{2}$ and $\sqrt{3}$, and irrational numbers, as these are called, were added to the number field. Later still, **negative numbers,** concerning which this chapter treats, were included. For hundreds of years negative numbers were rejected by mathematicians as impossible or meaningless but after a while it was seen that a negative result often has just as much meaning as a positive result and that negative numbers are necessary for the generalization and simplification of number work.

ILLUSTRATIVE EXERCISE

24. The following problems introduce negative numbers:

(1) A man can swim 3 miles an hour in still water. If his rate of swimming upstream is his rate in still water diminished by the rate of the current, how fast can he swim up a stream whose current is flowing 1 mile an hour?

Solution:

If x represents his rate upstream, we have,

$$x = 3 - 1 = 2$$

Therefore he can swim upstream 2 miles an hour.

(2) A man can swim 3 miles an hour in still water. How fast can he swim up a stream whose current flows 5 miles an hour? — 2 mi

Solution:

As before, let x represent his rate upstream. Then,

$$x = 3 - 5$$

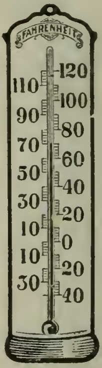
To solve the problem, we must subtract 5 from 3. That is, a number must be found, which, added to 5, gives 3. In arithmetic we should say that this is impossible, because there is no such number.

In problem 1, the man swims 2 miles an hour upstream.

In what direction does the man go in the second case? How fast does he go? 2 miles an hr

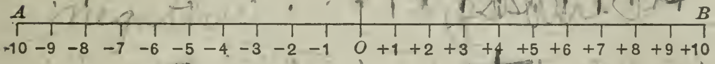
In the first case the man goes upstream and in the second case downstream. This is indicated by saying that in problem 1 his rate upstream is + 2 miles an hour and in problem 2 his rate upstream is − 2 miles an hour. Notice that the + and − signs, as here used, indicate an oppositeness in direction.

In the thermometer a certain point denoting a certain temperature is marked 0 and points above this indicate a higher temperature (warmer) and are considered positive. For example, a temperature of + 8° means 8° above the zero point, + 24° means 24 degrees above the zero point, etc. Points below the zero point indicate a lower temperature (colder) and are considered negative. For example, − 10° means 10° below the zero point. That is, the + and − applied to temperatures indicate an oppositeness in direction.



If we take a straight line and mark some point on it 0 and number the distances in each direction, 1, 2, 3, etc., we need, in locating a point, to know the direction as well as the distance from the zero point. If we regard distances to the right of the zero point as positive, then we may represent distances to the left as negative. Here, again, positive and negative numbers indicate oppositeness in direction. These illustrations suggest the following:

Negative numbers are the opposite of positive numbers.



If a positive number is used to represent a man's resources, his liabilities may be represented by a negative number. If distances north of the equator are called positive, then distances south of the equator are negative. If the buoyant force of a balloon is called positive, then a force pulling down, such as gravity acting on the ballast, is negative. Many other illustrations may be given.

If the following are represented by positive numbers, what are the corresponding negative numbers?

East longitude.
Time since the birth of Christ. — B. C.
Money received.
Distances measured upwards.
Increase in population.

25. Addition of Positive and Negative Numbers. Using the number series on line AB, start at zero and move a pencil point 4 spaces to the right, and then 2 more spaces to the right. How many spaces is the pencil point from zero? (6.) in what direction? (To the right.) That is,

$$+4+(+2)=+6$$

Start at zero and move a pencil point 5 spaces to the right and then 3 spaces to the left. How far is the pencil point from zero? in what direction? That is,

$$+5+(-3)=+2$$

Start at zero and move a pencil point 3 spaces to the left and then 5 spaces to the right. How far is the pencil point from zero? in what direction? That is,

$$-3+(+5)=+2$$

Start at zero and move a pencil point 7 spaces to the left and then 4 spaces to the right. How far is the pencil point from zero? in what direction? That is,

$$-7+(+4)=-3$$

Start at zero and move a pencil point 3 spaces to the left and then 5 spaces to the left. How far is the pencil point from zero? in what direction? That is,

$$-3+(-5)=-8$$

Start at zero and move a pencil point 10 spaces to the right, then 7 spaces to the left, and then 5 spaces to the right. How far is the pencil point from zero? 18

If a man's resources are \$5845 and his liabilities \$2675, his net worth is \$5845 + (− \$2675) = \$3170.

What is the result of moving a pencil point 7 spaces to the right of the zero point, then 4 spaces to the left, then 5 spaces to the left, then 3 spaces to the right? 1

How may we find the sum of two positive numbers? of two negative numbers?

How may we find the sum of two numbers, one of which is positive and the other negative? $8+(-3)=?$ 5
$3+(-7)=?$ 10

How may we find the sum of several numbers, some of which are positive and some negative? $(-3)+2+7+(-4)=?$

Bernice Manley

[illegible]

The **absolute value,** or **numerical value,** of a number is its value without regard to its sign. The absolute value of $+4$ is 4, and the absolute value of -4 is 4.

The sum of two positive numbers or two negative numbers is the sum of their absolute values, and has their common sign.

The sum of a positive number and a negative number is the difference between their absolute values and has the sign of the numerically greater number.

The sum of several numbers, some of which are positive and some negative, is the difference between the absolute value of the sum of the positive numbers and the absolute value of the sum of the negative numbers and has the sign of the numerically greater sum.

EXERCISES

Find the following sums, using the scale in section 24:

1. $+5+(-3)$ $+2$
2. $+6+(-2)$ $+4$
3. $-4+(-3)$ -7
4. $-5+(-2)$ -7
5. $-8+(+3)$ -5
6. $+7+(-4)$ $+3$
7. $-8+(+5)$ -3
8. $-5+(+8)$ $+3$
9. $-10+(+12)$ $+2$
10. $+8+(-7)$ $+1$
11. $+4+(+6)$ $+10$
12. $+7+(-7)$ 0
13. $-3+(+8)$ $+5$
14. $-5+(+14)$ $+9$
15. $+4+(-12)$ -8
16. $+9+(-7)$ $+7$
17. $-7+(-3)$ -10
18. $+9+(-4)$ $+5$
19. $+6+(-13)$ -7
20. $-9+(+16)$ $+7$
21. $-3+(-5)+(+6)$ -2
22. $+8+(-10)+(-3)+(+7)$ $+2$
23. $-9+(+7)+(+6)+(-12)$ -8
24. $+10+(-15)+(+3)+(-5)$ -7

25. What is the net result of a 10 dollar gain and a 7 dollar loss?

26. What is the net result of a 5 dollar gain and a 5 dollar loss?

Add $4a$ and $-8a$. (When no sign precedes a number $+$ is understood. $4a$ means $+4a$.)

According to the rule for adding numbers having a common factor, Section 17, we have

$$4a + (-8a) = [4 + (-8)]a = -4a$$

Begin

EXERCISES

Find the following sums:

1. $3a + 8a$ +11a
2. $17 + (-8)$ +9
3. $-18 + (+7)$ −11
4. $3abc + (-2abc)$ +1abc
5. $7k^2 + (-5k^2)$
6. $-3c + (-5c)$
7. $-12s^3 + (+14s^3)$
8. $-9 + (+6)$
9. $-24xy + (+14xy)$
10. $6x^2y + (-10x^2y)$
11. $15cs + (-cs)$
12. $8 + (-4) + (-5)$ −1
13. $5c + (+3c) + (-6c)$
14. $9x^2 + (-7x^2) + (+3x^2)$

15. Add $-12ac$, $4ac$, $3ac$, $21ac$, and $-5ac$.

16. Add $3bcx^2$, $-7bcx^2$, $-23bcx^2$, $17bcx^2$, and bcx^2.

17. Add $.7a^2x^2, -.3a^2x^2, .12a^2x^2, -.34a^2x^2$, and $2.3a^2x^2$.

18. Add $\frac{2}{3}mn^3$, $\frac{1}{2}mn^3$, and $-\frac{1}{3}mn^3$.

19. Find the sum of $2x(a+b)$, $-3x(a+b)$, $-4x(a+b)$, and $9x(a+b)$.

20. Find the sum of $\frac{13}{5}x^2$, $\frac{24}{5}x^2$, $-\frac{17}{5}x^2$, and $\frac{2}{5}x^2$.

Here

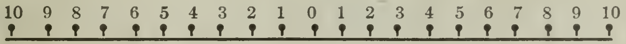
21. A man owns a house and lot worth +$3000 and has +$300 in the bank. He holds notes against other people for $400 and +$700. He owes −$900 in small items to merchants, and has given his note for −$800. What is he worth?

22. A merchant's inventory shows goods on hand worth + $5000; there are accounts due him for goods sold by him for + $1500; notes due him, + $500; he owes the wholesale houses − $1000 and owes on a note at the bank − $800. His house and lot are valued at + $3500. What is the merchant worth?

23. A man leaving a certain town goes 25 miles east the first day, 30 miles west the second day, 12 miles west the third day, and 24 miles east the fourth day. If the distances east are positive and those west negative, indicate by a positive or negative number the man's distance and direction from the town at the end of the fourth day.

24. A steamer starting in longitude − 30° west, runs − 4° west when the engines break down and the vessel is blown east − 2° from the point of the breakdown. With the engines repaired, the steamer goes − 5° west. Calling west longitude negative and east longitude positive, indicate, as in problem 23, the longitude of the steamer.

26. Subtraction of Positive and Negative Numbers. In the following problems each space on the scale represents one mile.



If one man is 8 miles to the right of the zero point and a second man 3 miles to the right of the zero point, how far is the first man from the second man? (5 miles.) in what direction? (To the right.) That is,

$$+8-(+3)=+5$$

If the first man is 8 miles to the left of the zero point and the second man 3 miles to the right of the zero point, how far is the first man from the second man? (11 miles.) in what direction? (To the left.) That is,

$$-8-(+3)=-11$$

If the first man is 2 miles to the right of the zero point and the second man 10 miles to the right of the zero point, how far and in what direction is the first man from the second? Write out the result as illustrated above.

If the first man is 8 miles to the left of the zero point and the second man is 9 miles to the right of the zero point, how far and in what direction is the first man from the second? Write out the result as above.

In these problems, subtracting a positive number gives the same result as adding a numerically equal negative number. $7-(+5)=7+(-5)$

If one man is 10 miles to the right of the zero point and a second man is 4 miles to the left of the zero point, how far is the first man from the second man? (14 miles.) in what direction? (To the right.) That is,

$$+10-(-4)=+14$$

If the first man is 8 miles to the left of the zero point and the second man is 5 miles to the left of the zero point, how far is the first man from the second man? (3 miles.) in what direction? (To the left.) That is,

$$-8-(-5)=-3$$

If one man is 6 miles to the right of the zero point and a second man is 6 miles to the left of the zero point, how far and in what direction is the first man from the second? Write out the result as illustrated above.

If one man is 5 miles to the left of the zero point and a second man 9 miles to the left of the zero point, how far is the first man from the second?

In these problems, subtracting a negative number gives the same result as adding a numerically equal positive number.

$$12 - (-4) = 12 + (+4)$$

We may, then, give this rule for subtraction:

Change the sign of the number in the subtrahend and proceed as in addition.

NOTE. The student should not be allowed to make the changes of sign on paper or on the board. He should, at the start, form the habit of making these changes mentally.

EXERCISES

Perform the following subtractions:

1\. $\begin{array}{r} +8 \\ +3 \\ \hline +9 \end{array}$ $\quad\begin{array}{r} -8 \\ -3 \\ \hline -5 \end{array}$ $\quad\begin{array}{r} -7 \\ -4 \\ \hline -3 \end{array}$ $\quad\begin{array}{r} +5 \\ -6 \\ \hline +11 \end{array}$ $\quad\begin{array}{r} +8 \\ +10 \\ \hline \end{array}$ $\quad\begin{array}{r} -20 \\ -32 \\ \hline \end{array}$ $\quad\begin{array}{r} +18 \\ -12 \\ \hline \end{array}$

2\. $\begin{array}{r} 7a \\ -3a \\ \hline +10 \end{array}$ $\quad\begin{array}{r} -8b^2 \\ 3b^2 \\ \hline \end{array}$ $\quad\begin{array}{r} 16k \\ -4k \\ \hline \end{array}$ $\quad\begin{array}{r} -8ab^2 \\ 3ab^2 \\ \hline \end{array}$ $\quad\begin{array}{r} 9bc \\ -10bc \\ \hline \end{array}$ $\quad\begin{array}{r} 4ak^2 \\ 7ak^2 \\ \hline \end{array}$

B. D

3\. $24x^2y - (+13x^2y)$ 5. $-15c^2d^2 - 27c^2d^2$

4\. $-32mnx^2 - (+17mnx^2)$ 6. $2.5\,cx^2 - 1.3\,cx^2$ 3.8

7\. $-3.8\,xz - (-2.1\,xz)$ x z

8\. $\frac{1}{2}x^2 - \frac{1}{3}x^2$ $\frac{5}{6}$

9\. $\frac{7}{3}c^2d - (-\frac{1}{2}c^2d)$ $= \frac{17}{6}$

10\. $2\frac{3}{7}x^2z^2 - (-1\frac{2}{7}x^2z^2)$ $+3\frac{5}{7}x^2z^2$

11\. $-34b^2r^3 - (-52b^2r^3)$ $+18$

12\. $11b^2ms^2 + (-4.5b^2ms^2)$ 15.5

13\. From the sum of $3x$ and $8x$ subtract the sum of $-11x$ and $6x$. 0

14\. From the sum of $-12mn$ and $-23mn$ subtract $(-25mn) - (-17mn)$. $+3mn$

15\. One place is 10° east longitude and a second place is 24° west longitude. How far is the first place east of the second? 34°

16. If the first place is 10° east longitude and the second place is 24° east longitude, how far is the first place from the second? 14°

17. Paris is 2° 20′ east longitude and New York is 73° 58′ west longitude. How far is Paris east of New York? 76 38′

18. The latitude of Rio de Janeiro is about 22° 50′ south and of Buenos Ayres about 34° 36′ south. How far is the latitude of Rio de Janeiro from the latitude of Buenos Ayres? 12° 14′

19. At noon the thermometer stood at + 10°. At midnight the reading was − 6°. What was the change in temperature? 16° ~~+6°~~

20. At 6 A. M. the thermometer reading was − 10°. From 6 A. M. to 9 A. M. it rose 7°; from 9 A. M. to noon it rose 12°; from noon to 3 P. M. it rose 6°; from 3 P. M. to 6 P. M. it fell 8°; and from 6 P. M. to midnight it fell 10°. What was the reading at midnight? 7°

Here

27. Negative Number an Unperformed Subtraction. To subtract 3 from 7, we find what number added to 3 will give 7. We cannot subtract 7 from 3, if we confine ourselves to positive numbers. Negative numbers were introduced to render this subtraction possible, and the number we add to 7 to give 3 is called a negative number; *viz.*, − 4. Negative numbers did not come into general use until Descartes (1596–1650) indicated the close connection existing between algebra and geometry.

In the above illustration, the − 4 may be regarded as an unperformed subtraction. In subtracting 7 from 3, the 7 may be separated into two parts, 3 and 4, and the subtraction written

$$3 - 7 = 3 - 3 - 4 = 0 - 4 = -4$$

1350

This leaves 4 not yet subtracted. Hence, the -4 may be thought of as 4 to be subtracted when there is something from which to subtract it.

28. Multiplication of Positive and Negative Numbers.

Note the following four cases that can arise in the multiplication of positive and negative numbers. The first is the multiplication of a positive number by a positive number. What is the second? the third? the fourth?

(1) $3 \cdot 5$ means that $+5$ is to be added 3 times, or $5 + 5 + 5 = 15$.

(2) $3 \cdot (-5)$ means that -5 is to be added 3 times, or $-5 + (-5) + (-5) = -15$.

(3) $-3 \cdot 5$ should give the opposite of $+3 \cdot 5$, which is $-3 \cdot (+5) = -15$

Or, we may think of this as 5 to be subtracted 3 times, *i. e.*, $-5 - 5 - 5 = -15$.

(4) $-3 \cdot (-5)$ should give the opposite of $3 \cdot (-5)$, which is $-3 \cdot (-5) = +15$

Or we may think of this as -5 to be subtracted 3 times, which, as we have seen, is the same as adding $+5$ three times.

The **law of signs for multiplication**, illustrated by the above, is as follows:

When the multiplier and multiplicand have like signs, the product is positive. When the multiplicand and multiplier have unlike signs, the product is negative.

It follows from this law that the product of an odd number of negative factors is negative and the product of an even number of negative factors is positive.

Like signs [illegible] +
Unlike signs [illegible] –

EXERCISES

Perform mentally the following multiplications:

1. $3 \cdot 5$
2. $3 \cdot (-5)$ –15
3. $4 \cdot (-7)$ – [illegible]
4. $-6 \cdot (+7)$ – [illegible]
5. $-6 \cdot (-7)$ + [illegible]
6. $-5 \cdot (-8)$ + [illegible]
7. $3 \cdot 3a$ +9a
8. $4 \cdot (-5x)$ – [illegible]
9. $-6 \cdot (-2y^2)$ + [illegible]
10. $-8 \cdot (-3ab)$ + [illegible]
11. $-7 \cdot (+4cx^2)$ – [illegible]
12. $+8 \cdot (-7cy^2)$ + [illegible]
13. $5 \cdot 6 \cdot (-3a)$
14. $(-5) \cdot (4) \cdot (-6x^2)$
15. $-5 \cdot 3a \cdot (-2b)$
16. $-6a \cdot (-2x) \cdot (-3y)$
17. $-2x \cdot (-3z) \cdot (+4c)$
18. $-4a \cdot (-6b) \cdot (-10c)$
19. $-3xy \cdot (-5a) \cdot (-2)$
20. $-7.5m \cdot 2.4x$
21. $\frac{2}{3}rs \cdot (-3) \cdot 4t$
22. $-8 \cdot (-6)$
23. $-5 \cdot (+3ay)$
24. $3a \cdot (-6by)$
25. $-7 \cdot (-6a) \cdot 2b$
26. $-8 \cdot (-3) \cdot 2.5 \cdot (-4)$

Begin Mon

29. Division of Positive and Negative Numbers.

From (1) under multiplication, (section 28), it is evident that $+15$ divided by $+3$ gives $+5$. That is,

$$+15 \div (+3) = +5$$

From (3), (Section 28), -15 divided by $+3$ gives -5. That is,

$$-15 \div (+3) = -5$$

From (2), (Section 28), -15 divided by (-3) gives $+5$. That is,

$$-15 \div (-3) = +5$$

From (4), (Section 28), $+15$ divided by (-3) gives -5. That is,

$$+15 \div (-3) = -5$$

12 16 32 16

16
12
28

These statements illustrate the following law of signs for division:

When the dividend and divisor have like signs, the quotient is positive, and when they have unlike signs, the quotient is negative.

$9a^2 + 3$

EXERCISES

Perform mentally the following divisions:

1. $6 \div 3$ $+2$
2. $9 \div (-3)$ -3
3. $8 \div (-2)$ -4
4. $-24 \div (+6)$ -4
5. $-36 \div (-9)$ $+4$
6. $4ab \div 2b$ $2a$
7. $6abc \div (-2c)$ -3
8. $-25xyz \div (-5xy)$ $+5z$
9. $-42x^2y \div (+6x^2)$ $-7y$
10. $56mn \div (-8n)$ $-7m$
11. $-70pq \div (+10pq)$ -7
12. $28a^3bc \div (-7bc)$
13. $-45xy^2 \div (-15y^2)$
14. $25ghk \div (+5h)$
15. $-27r^2s \div (-3r^2)$
16. $63x^2y \div (-21x^2y)$
17. $-23mn \div mn$
18. $-105x^3z \div (35x^3z)$
19. $4.5m \div 1.5$
20. $-3.2x^2y \div .8x^2$
21. $\frac{4}{7}x \div 4$
22. $\frac{15}{4}x^2y \div (-5y)$
23. $\frac{2}{7}mn^2 \div (-3m)$
24. $-.25m(a+b) \div (-.5m)$
25. $-6.4x^2(m-n) \div 1.6(m-n)$

30. Averages.

The sum of several numbers divided by the number of numbers is called the **average** of those numbers.

The average of 12, 4, 7, and 9 is

$$\frac{12+4+7+9}{4} = \frac{32}{4} = 8$$

What is the average of 14, −10, −5, 24, and 8?

$$\frac{14+(-10)+(-5)+24+8}{5} = \frac{31}{5} = 6\tfrac{1}{5}.$$

The average of two numbers is called their **arithmetic mean.**

If the highest temperature at a certain place on a certain day was 10° above zero and the lowest 8° below zero, the mean temperature for that day was

$$\frac{10° + (-8°)}{2} = 1°$$

EXERCISES

1. Find the mean of
   (a) 35° above zero and 55° above zero.
   (b) 40° above zero and 20° below zero.
   (c) $-12°$ and $+16°$
   (d) $+46°$ and $+60°$
   (e) $-24°$ and $-15°$

2. Find the average of
   (a) 12, -24, 25, 32, and -15.
   (b) 25, 10, -34, and -13.
   (c) 2.5, -3.6, -1.4, and 3.7.
   (d) $12x$, $4x$, $-20x$, and $-8x$.

3. The mean temperatures for a week at a certain place were: Monday 30°, Tuesday 45°, Wednesday 60°, Thursday 54°, Friday 62°, Saturday 75°, and Sunday 80°. What was the average mean temperature for the week?

4. The precipitation (rain, snow, etc.) by months at a certain station of the Weather Bureau was as follows:

January, 2 in.	July, 3.4 in.
February, 1.5 in.	August, 2.1 in.
March, 2.5 in.	September, 3.8 in.
April, 3.2 in.	October, 2.3 in.
May, 2.8 in.	November, 1.2 in.
June, 2.4 in.	December, .8 in.

What was the average monthly precipitation at the given station?

5. A business report shows the following for four months: January, gain of $500; February, gain of $400; March, loss of $350; April, gain of $150. What was the average monthly gain or loss?

REVIEW EXERCISES

31. 1. Why are negative numbers necessary in algebra? Give a problem to illustrate your answer.

2. Add 4, -3, 7, and -12. — 4

3. Find the sum of $-2ac$, $7ac$, $-6ac$, and $-3ac$. −4a

4. Define negative numbers. Give illustrations.

5. State the rules for adding positive and negative numbers.

6. State the rule for subtracting positive and negative numbers.

7. $-12-(+3)=?$ −15 $+24-(+35)=?$
$-11-(-5)=?$

8. If the following are represented by positive numbers, what do the corresponding negative numbers represent?

- (a) Distances east of Chicago.
- (b) Distances north of St. Louis.
- (c) Business gains.
- (d) Money earned.
- (e) North latitude.
- (f) East longitude.
- (g) Number of degrees above zero.
- (h) A man's property.
- (i) A note due me.

9. State the law of signs for multiplication of positive and negative numbers.

10. State the law of signs for division of positive and negative numbers.

11. What is the product of $12mn$ and $-3a$? $-6acd$ and $-15b$?

12. $(-32xy^2) \div 8x = ?$ $(-2.5mn) \div (-5m) = ?$

13. What number added to 12 gives 4? What number subtracted from 3 gives 8?

14. Add $+12ax$, $-3ax$, $-23ax$, $7ax$, and $-15ax$.

15. The sum of two numbers is -9 and one is twice the other. What are the numbers?

16. In the formula $C = \frac{5(F - 32)}{9}$, find the value of C, if F = 20.

17. $3a^2 - 11a^2 + \frac{24a^2x}{8x} = ?$

18. What is the value of 2 $x^2 - 3x + 6$, if $x = -2$? if $x = -3$? if $x = 4$?

19. From the sum of $2mnx^2$, $-16mnx^2$, and $23mnx^2$, subtract the product of $-3mn$ and $-7x^2$.

20. Evaluate the following if $a = -2$, $x = -3$, and $y = 2$:

(a) $4ax - 3xy + 2y$.

(b) $(x + 2y)^2 - \frac{4ax}{y} + \sqrt{3axy}$.

21. Express in words: $(x^2 - y^2) \div (x + y) = (x - y)$. Verify if $x = 10$ and $y = 4$; if $x = 5$ and $y = 11$.

22. What is the average weight of five men weighing as follows: 135 lb., 250 lb., 178 lb., 185 lb., and 150 lb.?

23. Divide $4 \cdot 10 \cdot 8$ by -4, in three ways.

24. Multiply $3 \cdot 4 \cdot 7$ by -3, in three ways.

25. A railroad train runs 30 miles an hour. How far will it run in 10 hours? in c hours?

26. If a man walks at the rate of 3 miles an hour, how far will he walk in 2 hours? in x hours?

27. If d represents the distance, r the rate, and t the time, write a formula showing how any problem can be solved in which the rate and time are given and the distance is required.

28. Make and solve two problems like the preceding and state what numbers are represented by d, r, and t in each problem.

29. A boatman rows 6 miles an hour for 2 hours across a lake and 8 miles an hour for 3 hours down a stream. What is his average rate of rowing per hour?

30. A farmer has 40 acres of corn, 30 acres of wheat, and 20 acres of oats. His corn yielded 1600 bushels, his wheat 450 bushels, and his oats 740 bushels. What was the average yield of each grain per acre?

SUMMARY

SECTION

23. Extension of negative numbers.
24. Illustrations of negative numbers.
25. Addition of positive and negative numbers. Problems.
26. Subtraction of positive and negative numbers. Problems.
27. Negative number an unperformed subtraction.
28. Multiplication of positive and negative numbers. Law of signs. Problems.
29. Division of positive and negative numbers. Law of signs. Problems.
30. Averages. Problems.
31. Review exercises.

[illegible]
Write [illegible]
[illegible]
[illegible] in class

21
24
[illegible]

CHAPTER IV

ADDITION

32. Addition of Monomials.

A **monomial** is an algebraic expression of one term.

A **polynomial** is an algebraic expression of two or more terms.

A **binomial** is a polynomial of two terms.

A **trinomial** is a polynomial of three terms.

EXERCISES

Name the monomials, binomials, and trinomials in the following expressions:

1. $a - x$
2. $3ax^2y$
3. $m - 7n + 3x^2$
4. $2(2 - b) + c^2$
5. $3ax + 4b^2c$
6. $12am(x + y)$
7. $\frac{ax}{3y} + m^2$
8. $2xy - 3m^2(r + s)$
9. $m + 2n - \frac{x}{5}$

ILLUSTRATIVE EXERCISE

Add 32, 56 and 24.

Solution:

$$\begin{aligned} 32 &= 4\cdot 8 \\ 56 &= 7\cdot 8 \\ 24 &= \underline{3\cdot 8} \\ & \ 14\cdot 8 = 112 \end{aligned}$$

We have here, in accordance with the principle laid down in section 17, added the coefficients of the common factor, 8, and multiplied 8 by the sum of its coefficients.

The student may check by adding the numbers in the usual way.

EXERCISES

Add the following sets of numbers by this method:

$5\cdot 9$	$32 = 2\cdot 16$	$35 = 5\cdot 7$	81	333	54	25
$4\cdot 9$	$48 = 3\cdot 16$	$49 = 7\cdot 7$	15	777	36	40
$3\cdot 9$	$80 = 5\cdot 16$	$84 = 12\cdot 7$	27	222	72	95
$6\cdot 9$	$16 = 1\cdot 16$	$14 = 2\cdot 7$	24	110	42	75
		$26\cdot 7$		22		

ILLUSTRATIVE EXERCISE

Add $5bc$, $3bc$, $7bc$, and $14bc$.

Solution:

Adding the coefficients of the common factor gives 29. Hence the sum is $29bc$.

Check: Let $b = 1$ and $c = 2$. Then

$$\begin{aligned} 5bc &= 5\cdot 1\cdot 2 = 10 \\ 3bc &= 3\cdot 1\cdot 2 = 6 \\ 7bc &= 7\cdot 1\cdot 2 = 14 \\ \underline{14bc} &\underline{= 14\cdot 1\cdot 2 = 28} \\ 29bc &= 29\cdot 1\cdot 2 = 58 \end{aligned}$$

Substituting these results in $5bc + 3bc + 7bc + 14bc = 29bc$, we have

$$10 + 6 + 14 + 28 = 58$$

This tells us that our result is probably correct.

EXERCISES

Perform the following additions, mentally when possible:

1. $4x + 7x + 3x + 12x$

2. $6yz + 12yz + 7yz$

3. $9k + 14k + 2k + k$

4. $11m^2 + 4m^2 + 9m^2$

5. $8a + 3a + 12a + 7a$

6. $9bc + 10bc + 12bc$

7. $7a^2b + 5a^2b + 8a^2b$

8. $13xy + 5xy + 17xy$

9. $5x^2y + 9x^2y + 13x^2y$

10. $32ab + 7ab + 11ab$

11. $7gh^2 + 13gh^2 + 20gh^2$

12. $8x^2y^2 + 17x^2y^2 + 12x^2y^2$

13. $4cm^3 + 23cm^3 + 8cm^3$

14. $5x^2y^2z + 7x^2y^2z + 12x^2y^2z$

15. $4(a + b) + 9(a + b)$

16. $\frac{1}{7}c + \frac{2}{7}c + \frac{3}{7}c$

17. $\frac{1}{5}s + \frac{2}{5}s + \frac{4}{5}s$ $\frac{7}{5}s$

18. $\frac{1}{2}n + \frac{2}{3}n + \frac{1}{4}n + \frac{1}{3}n$ $\frac{21}{12}n$

19. $\frac{1}{2}b + 2b + \frac{1}{3}b + \frac{1}{6}b$

20. $6x(y + z)^2 + 12x(y + z)^2$

We have seen (section 25) that $-7 + (-3) = -10$. Hence, $-7a + (-3a) = -10a$ and $-7 \cdot 6 + (-3 \cdot 6) = -10 \cdot 6 = -60$.

Find the following indicated sums, mentally when possible:

21. $-3x + (-4x) + (-6x)$

22. $-2y + (-3y) + (-6y)$

23. $-3xz + (-7xz) + (-8xz)$

24. $-45 + (-20) + (-35)$

25. $-21 + (-49) + (-56)$

26. $-2 \cdot 5 \cdot 6 + (-7 \cdot 5 \cdot 6) + (-3 \cdot 5 \cdot 6)$

27. $-7bc + (-5bc) + (-3bc)$

28. $-12z^2 + (-4z^2) + (-5z^2)$

29. $-13bxy + (-7bxy) + (-9bxy)$

30. $-14cm^2 + (-6cm^2) + (-8cm^2)$

31. $-7xyz + (-8xyz) + (-9xyz) + (-xyz)$

32. $-12wz^2 + (-3wz^2) + (-4wz^2) + (-10wz^2)$

33. $-8amx + (-17amx) + (-11amx) + (-4amx)$

34. $-3x^2y^2 + (-2x^2y^2) + (-7x^2y^2) + (-x^2y^2)$

35. $-2(a + b) + [-3(a + b)] + [-4(a + b)]$

36. $-6(x + y)^2 + [-5(x + y)^2] + [-2(x + y)^2]$

37. $-\frac{1}{2}x + (-\frac{1}{3}x) + (-\frac{1}{6}x)$

38. $-\frac{2}{5}a + (-\frac{1}{2}a) + (-\frac{1}{5}a)$

39. $-\frac{1}{6}m + (-\frac{1}{3}m) + (-\frac{5}{6}m)$ $-1\frac{1}{3}m$

40. $-\frac{1}{4}b + (-\frac{2}{3}b) + (-\frac{5}{8}b)$

33. Laws of Order and Grouping. In arithmetic we have found that $4 + 3 = 3 + 4$, and that $2 + 5 + 3 = 2 + 3 + 5 = 3 + 2 + 5 = 5 + 2 + 3 = 5 + 3 + 2 = 3 + 5 + 2$.

In algebra it is assumed that

$$a + b = b + a$$

and $a + b + c = a + c + b = b + a + c = b + c + a$, etc., whatever the values of a, b, and c. That is,

The sum of several numbers is the same in whatever order they are added.

This is called the **commutative law of addition.**

It is evident that

$3 + 5 + 6 + 2 = (3 + 5) + (6 + 2) = (3 + 5 + 6) + 2$, etc.

In algebra it is assumed that

$a + b + c + d = (a + b) + (c + d) = (a + b + c) + d$, etc. That is,

Numbers to be added may be grouped in any manner without altering the sum.

This is called the **associative law of addition.**

34. Addition, continued. We saw (section 25), that $+7 + (-3) = +4$ and $-7 + (+3) = -4$.

Likewise $7a + (-3a) = [7 + (-3)]a = 4a$.

ILLUSTRATIVE EXERCISES

1. Find the sum of $5a$ and $-2a$.

Solution:

$$5a + (-2a) = [5 + (-2)]a = 3a$$

2. Find the sum of 35 and -25.

Solution:

$$35 + (-25) = [7 + (-5)]\,5 = 2 \cdot 5 = 10$$

This is conveniently arranged as follows:

$$\begin{array}{rcr} 35 & = & 7\cdot 5 \\ -25 & = & -\underline{5\cdot 5} \\ & & 2\cdot 5 = 10 \end{array}$$

EXERCISES

Find in this way the following sums:

1. $\begin{array}{r} -3\cdot 24 \\ \underline{2\cdot 24} \end{array}$

2. $\begin{array}{r} -35 \\ \underline{60} \end{array}$

3. $\begin{array}{r} 42 \\ \underline{-24} \end{array}$

4. $\begin{array}{r} 39 \\ \underline{-52} \end{array}$

5. $\begin{array}{r} -63 \\ \underline{49} \end{array}$

6. $\begin{array}{r} 120 \\ \underline{-\ \ 84} \end{array}$

7. $8x + (-5x)$
8. $12x + (-15x)$
9. $17bc + (-10bc)$
10. $20bc^2 + (-32bc^2)$

Check problems 9 and 10 for $b = 1$ and $c = 2$.

11. $16xy + (-9xy)$
12. $4xy + (-3xy)$
13. $-2z^2 + 6z^2$
14. $13w^2 + (-20w^2)$
15. $-11ad + 3ad$
16. $13c^2k + (-8c^2k)$
17. $-26x^2y^2 + 31x^2y^2$
18. $m^2 + (-12m^2)$
19. $5a^2b + (-10a^2b)$
20. $-3(a+b)+5(a+b)$
21. $\frac{3}{4}ab + (-\frac{1}{2}ab)$
22. $\frac{3}{4}c^2d^2 + (-\frac{4}{3}c^2d^2)$

Since the order in which the terms are added is immaterial (see section 33, commutative law), when there are more than two terms, some of which are positive and some negative, they may be added in order, or the positive terms and the negative terms may be combined separately and the sum of these results found. For example:

$6 + (-3) + (-5) + 7 = 6 + 7 + (-3) + (-5) = 13 + (-8) = 5$

$7a + (-3a) + 4a + (-8a) + (-5a) = 7a + 4a + (-3a) + (-5a) + (-8a) = (7+4)a + [(-3) + (-5) + (-8)]a = 11a + (-16a) = -5a$

63/1 17

EXERCISES

Find the following sums:

1. $7a + (-3a) + 8a + 5a$ $+17a$
2. $4m + (-5m) + (-7m) + (-12m)$ $-20m$
3. $(-10b) + 4b + 12b + (-18b)$ $-12b$
4. $6ax + (-12ax) + (-3ax) + 4ax$ $-5ax$
5. $(-8p^2) + (-7p^2) + (-4p^2) + (-11p^2) + 17p^2$ $15p$

Add the following:

6. $+3$, $+4$, -7, -6, and $+8$ $+2$
7. $5c$, $-7c$, $4c$, and $8c$ $+10c$
8. $17k$, $-3k$, $-7k$, and $-4k$ $-3k-0$
9. $-12ac$, $-3ac$, $4ac$, and $-2ac$ $-13ac$
10. $3s^2$, $-6s^2$, $-8s^2$, and $12s^2$ $+1s^2$
11. $4(a+b)$, $3(a+b)$, $-8(a+b)$ and $5(a+b)$ $+4$
12. $5xyz$, $-11xyz$, $18xyz$, $-4xyz$, and $-13xyz$ $-$
13. $4a^2bc$, $-2a^2bc$, $10a^2bc$, and $-18a^2bc$ $-6a^2bc$
14. $10m^2n^2$, $-2m^2n^2$, $-5m^2n^2$, $16m^2n^2$, and $-8m^2n^2$ $+17$
15. $6(x+y)^2$, $-12(x+y)^2$, $4(x+y)^2$, and $-11(x+y)^2$ $-13(x+y)^2$
16. $\frac{1}{2}x$, $\frac{1}{4}x$, $-\frac{3}{8}x$, and $\frac{7}{8}x$
17. $\frac{3}{5}x$, $-\frac{4}{5}x$, $-\frac{1}{10}x$, and $\frac{3}{10}x$
18. $\frac{2}{3}a^2b$, $-\frac{3}{4}a^2b$, $\frac{1}{2}a^2b$, and $\frac{1}{8}a^2b$
19. $.4(a-m)$, $.7(a-m)$, and $-.2(a-m)$

If the coefficients of the common factor are not definite numbers, the addition of the coefficients must be indicated. If the sum of ax and bx is desired, the coefficients, a and b, of the common factor, x, cannot be added as we add 3 and 4, but the addition is indicated. Thus,

$$ax + bx = (a + b)x$$

ILLUSTRATIVE EXERCISE

Add $2ax$ and $-3cx$

Solution:

$2ax + (-3cx) = [2a + (-3c)]x = (2a - 3c)x$

Add the following:

20. $9ax$ and $3bx$

21. $4pxy$ and $3qxy$

22. $7as^2$ and $-4cs^2$

23. $3bc$, $7bc$, and $-5bc$

24. $4ax^2$, $3bx^2$, and $7ax^2$

25. $5ck$, $-3ak$, and $4bk$

26. $3(a+b)$ and $8(a+b)$

27. $7(x-y)$ and $4(x-y)$

28. $4r(a+b)$ and $3x(a+b) = (4r+3x)(a+b)$

29. $5a(x+y)$ and $4c(x+y)$

30. $3(x+z)$, $5(x+z)$, and $3c(x+z)$

35. Addition of Polynomials. (Common Num[illegible])

Add $3a + 2b + c$, $4a - 3b - c$, and $2c - 8a + 6b$

Since numbers may be added in any order, we may first add the terms containing a, then those containing b, and then those containing c, and indicate the sum of the numbers so found.

$(3a + 2b + c) + (4a - 3b - c) + (2c - 8a + 6b) =$
$3a + 4a + (-8a) + 2b + (-3b) + 6b + c + (-c) + 2c.$
$3a + 4a + (-8a) = (3 + 4 - 8)a = -a$
$2b + (-3b) + 6b = (2 - 3 + 6)b = 5b$
$c + (-c) + 2c = (1 - 1 + 2)c = 2c$

Hence, the sum is $-a + 5b + 2c$

This work is done most conveniently by writing the terms in columns as follows:

$$\begin{array}{rr}
3a + 2b + c & 6 \\
4a - 3b - c & 0 \\
-8a + 6b + 2c & 0 \\
\hline
-a + 5b + 2c & 6
\end{array}$$

We may check by letting $a = 1$, $b = 1$, and $c = 1$, and writing the results at the side as above.

The terms $3a$, $4a$, and $8a$ in the first column are called similar terms. How do they differ?

Like or **similar terms** are terms that are alike or that differ in their coefficients only.

The first step in the addition of polynomials is to place the similar terms in columns. Each column is a problem in addition of monomials.

EXERCISES

Add the following polynomials and check results:

1. $2a + 3b$, $5a + b$, $6a + 2b$
2. $2x + y + 3z$, $x + 4y + z$, and $5x + 6y + 5z$
3. $7c - 4k$, $2c + 3k$, and $c - 5k$
4. $6m + 9n + 2p$, $7m - 10n - 3p$, and $4m - 5n - 8p$
5. $3r + 2s$, $5r - 3s$, and $-2r + 7s$
6. $a - 4b + 8c$, $7a - b - 8c$, and $8a + 5b - 6c$
7. $2d^3 + e^3 + 4$, $4e^3 + 3d^3 - 7$, and $3 + 7e^3 + 2d^3$
8. $7x^2 + 3x - 10$, $3x^2 - 2x + 4$, and $-3x + x^2 - 1$
9. $6abx - 7ghz + cdy$, $2abx - 4cdy + 3ghz$, and $4cdy - 8abx$
10. $4ax + 3my + c^2d$ and $2ax + 4my - 7c^2d$

11. $8a - 6b - 4c,\ 3c - 4a + 2b$, and $5b + 3c$

12. $x^2 + y^2 + z^2,\ 3x^2 + 4y^2 + 5z^2$, and $-2x^2 - 5y^2 + z^2$

13. $5x^2 + 4xy + 3y^2,\ 2x^2 - 5xy + 6y^2$, and $3x^2 - 5y^2 + 2xy$

14. $5x^3 + 3x^2 - 3x - 8,\ x^2 - 3x^3 + 31,\ 2x^3 - 8x - 5x^2 + 2$, and $4x + 7x^2 - 9 - 3x^3$

15. $4ab^2 + 4cde - 6h^2,\ 2ab^2 + 3h^2,\ 9cde - 7ab^2$, and $7ab^2 + 2cde - 5h^2$

16. $a^3 + 5ab^2 + b^3,\ a^3 - 10ab^2 + 5b^3$, and $5ab^2 - 2b^3 - 2a^2b$

17. $12y - 5a - 7ax,\ 5ax - a - 8y,\ 9a - y - 4ax,\ 4ax - 3a + 5y$, and $y + a - ax$

18. $\frac{1}{2}a + \frac{1}{3}b + \frac{1}{4}c,\ \frac{3}{4}a + \frac{1}{6}b + \frac{1}{2}c$, and $\frac{2}{3}a + \frac{1}{4}b + \frac{3}{4}c$

19. $\frac{2}{5}xy - \frac{2}{3}mn - \frac{1}{3}p,\ \frac{1}{2}p + mn - \frac{3}{10}xy$, and $\frac{1}{6}mn + \frac{3}{4}p$

20. $ax + by + cz,\ mx - ny + pz$, and $hz - kx - ly$

Solution:

$$\begin{array}{rrr} ax + & by + & cz \\ mx - & ny + & pz \\ -\ kx - & ly + & hz. \\ \hline \end{array}$$

$$(a + m - k)x + (b - n - l)y + (c + p + h)z$$

21. $bl + cm + dn$ and $bx - cy - dz$

22. $2ax + 3by + 5cz$ and $3mx - 6nz + 8ry$

23. $3a(x + 6) + 2(y + 4)$ and $5(x + 6) - 4b(y + 4)$

24. $3x + 8x - 5x$ are how many x? $ax + cx$ are how many x? $ac - bc + 3c$ are how many c? $c(x + y) + 3(x + y)$ are how many $(x + y)$?

25. Solve the equation $7x + 2x + 3x = 36$.

26. Solve the equation $3x + 2x + x = 18$.

27. The sum of three numbers is 56. The second number is twice the first and the third is 5 times the first. What are the numbers?

28. A rectangular field is twice as long as wide and its perimeter is 240 rods. Find the length and the width of the field.

29. In a certain school the number of boys is one-half the number of girls. How many are there of each if the total number of pupils is 240?

REVIEW EXERCISES

36. **1.** What is the sum of $2a^2 + 3ay + 4y^2$, $2y^2 - 4a^2 - 3ay$, and $3a^2 - 2ay - 3y^2$?

2. A railroad 130 miles in length is increased by m miles. What is its length?

3. What is the total cost of a yards of cloth at c cents per yard and of b yards at d cents per yard?

4. Add $7as$, $-3bs$, and $2cs$.

5. What is the sum of three consecutive numbers the smallest of which is x? $x+2$

6. What is the sum of three consecutive odd numbers the smallest of which is y?

7. What is the sum of three consecutive numbers if the middle number is a?

8. If a train averages c miles per hour, how far will it go in k hours?

9. Express m gallons and b quarts in quarts.

10. Add $3ax^2 + 2ay + cd$, $4cd - 2ax^2 - 7ay$, and $7ay - 2cd + 3ax^2$. Check the result.

11. The weights of the members of a basket-ball team are 135 lb., 152 lb., 167 lb., 175 lb., and 161 lb. What is their average weight?

12. The batting averages of the members of a base ball team were .250, .302, .264, .242, .287, .325, .276, .290, and .194. What was the batting average of the team?

13. Solve: $8x + 2x = 12 - 10 + 8$.

14. Separate 77 into 3 parts such that the first part shall be twice the second part and four times the third part.

15. A base ball team played 21 games. The number of games won was twice the number of games lost. How many games were won and how many lost?

16. What number is c more than m?

17. Write a number 2 more than twice the sum of the numbers x and y.

18. A man paid x dollars for a wagon, y dollars for a harness, and for a horse, z dollars more than he paid for wagon and harness. What did the horse cost him?

19. Solve: $7x + 8x - 11x + 6x = 22 - 3 + 11$.

20. Separate 96 into three parts such that the first is 3 times the second, and the third equal to the sum of the other two parts.

4

SUMMARY

SECTION
32. Definitions. Addition of monomials. Examples.
33. Law of order. Law of grouping.
34. Addition of monomials continued. Examples.
35. Addition of polynomials. Examples.
36. Review exercises.

CHAPTER V

AXIOMS AND EQUATIONS

37. Axioms. The solution of equations more complicated than those in section 1 requires certain assumptions. These assumptions are called **axioms.**

Axiom I. If two numbers are equal to the same number or to equal numbers, they are equal to each other.

$4 + 5 = 9$ and $3 + 6 = 9$. Therefore, $4 + 5 = 3 + 6$.

If $a + b = x$ and $c + d = x$, then $a + b = c + d$.

If $x = a$ and $y = a$, then $x = ?$ If $x = y$ and $5 = y$, what does the axiom tell us?

Axiom II. If equals are added to equals, the sums are equal.

If $a = 5$ and $b = 3$, then $a + b = 5 + 3 = 8$.

If $x = 8$, then $x + 3 = 8 + 3 = 11$.

If $x = 5$ and $a = 2$, what does $x + a$ equal? $x + 5 - 5\ 10$

Axiom III. If equals are subtracted from equals, the remainders are equal.

If $a = 5$ and $b = 3$, then $a - b = 5 - 3 = 2$.

If $x = 8$, then $x - 3 = 8 - 3 = 5$. $y - 3 = 7 - 5 = 3$

If $x = 5$, what is obtained by subtracting 5 from each of the equals?

Axiom IV. If equals are multiplied by equals, the products are equal.

If $c = 3$, then $3c = 9$.

If $c = 4$ and $b = 5$, then $c \cdot b = 4 \cdot 5 = 20$.

If $a = 5$ and $x = 6$, what does ax equal?

$a = 5\ x = 6$

$a . x$ or $5 . 6 = 30$

Axiom V. If equals are divided by equals, the quotients are equal.

If $5k = 15$, then $k = \frac{15}{5} = 3$

If $ac = bc$, then $\frac{ac}{c} = \frac{bc}{c}$, or $a = b$

If $a = 6$ and $x = 3$, what does $\frac{a}{x}$ equal? $\frac{6}{3}$

Axiom VI. Like powers of equals are equal and like roots of equals are equal.

If $a = 3$, then $a^2 = 9$, and $a^4 = 81$

If $x^2 = 25$, then $x = 5$

If $y^3 = 8$, what does y equal?

38. Equations. An equation expresses the equality of two numbers. (See section 46.) The members of an equation are the parts on the two sides of the equality sign. That on the left side is called the **first member,** or the **left hand member;** that on the right side, the **second,** or **right hand member.**

Since the members of an equation are equals, it is evident that axioms II–VI apply to the members of an equation. Therefore,

The same number may be added to each member of an equation or subtracted from each member of an equation without destroying equality.

Each member of an equation may be multiplied or divided by the same number without destroying equality.

39. Solution of Equations. In solving an equation the letter whose value is sought is often called the **unknown.** A number which, when substituted for the unknown, makes the two members of the equation equal is said to **satisfy** the equation and is called a **root,** or **solution,** of the equation.

In the equations already solved, all the terms containing the unknown have been in one member and all the remaining terms in the other member. In each case the equation was solved by collecting the terms, when necessary, and dividing both members by the coefficient of the letter whose value was sought.

ILLUSTRATIVE EXERCISES

1 Solve the equation

$$7x - 2x = 10 + 5$$

Solution:

Collecting terms, $5x = 15$

Dividing by 5, $x = 3$

When the terms containing the unknown are not all in one member of the equation and the known in the other, we may add equal amounts to both members or subtract equal amounts from them, using whatever amount is necessary to bring the equation into this form.

2. Solve: $2x + 3 = 8$

Solution:

$$2x + 3 = 8$$

Adding -3 to each member (Axiom II),

$$\underline{\quad -3 = -3}$$

we have $2x = 8 - 3$, or 5

Dividing each member by 2 (Axiom V), we have $x = 2\frac{1}{2}$

Check: $2 \times \frac{5}{2} + 3 = 8$

Since $\frac{5}{2}$, substituted for x in the original equation, satisfies the equation, that is, makes the two members equal, $\frac{5}{2}$ is a root, or solution, of the given equation.

3. Solve: $5x - 3 = 3x + 9$

Solution: $5x - 3 = 3x + 9$

Adding 3 to each member (Axiom II), $+3 = +3$

we have $5x = 3x + 12$

Adding $-3x$ to each member (Axiom II), $-3x = -3x$

we have $2x = 12$

Dividing each member by 2 (Axiom V), we have $x = 6$

Check: $5 \cdot 6 - 3 = 3 \cdot 6 + 9$

$27 = 27$

Since 6, substituted for x in the original equation, satisfies the equation, 6 is a root, or solution.

EXERCISES

Solve the following equations and check the results in each case by substituting in the original equation the value obtained for the unknown. Solve the simpler equations mentally.

1. $5x = 20$
2. $3x = 12$
3. $2x = -8$
4. $x + 5 = 8$
5. $2x + 1 = 17$
6. $2x - 5 = 7$
7. $2x + 3x - 6 = 9$
8. $8x - 3x + 2 = 17$
9. $6x - 15 = -6$
10. $4x - 2 = 3x + 4$
11. $2x - 6 = 9 + 5x$
12. $2x - x = 6 - x$
13. $3y - y = 10 + y$
14. $7s - 8 = 12 + 3s$
15. $7 - y = -8 + 2y$
16. $12 + 2m + 3 = 5m$
17. $17 + 10 - 15x = 18$
18. $\frac{2}{3}n + 2 = 10 - \frac{1}{3}n$
19. $3x + 4x = 27 - 2x$
20. $3r - 2 = r + 10$
21. $5x + 3x = 2x + 18$
22. $3x - x - 6 = 18 - x$

23. $18 - 2s = s + 6$

24. $3a + 6a - 10 = 4a + 15$

25. $3 + 7a = 2a + 13$

26. $3x + 15 = 12x + 12$

27. $6k + 18 - 4k = 9k - 3$

28. $4b + 20 = 10 - 3b$

29. $5x - 2x = 30 + x$

30. $3z - 7 + 2z = 14 - 2z$

31. $10x - 6x = 16 + 84$

32. $10 + 2 \cdot 5x = x - 4$

In solving problems the following suggestions may be of service:

(a) Read the problem thoughtfully.

(b) Decide what the letter used shall represent. This will usually be the number or one of the numbers sought.

(c) Find two expressions for the same number and connect them by the sign of equality.

(d) Solve the resulting equation.

33. In one school there are 7 more students than in a second school. In both there are 277 students. How many are there in each school?

Suggestion:

If x represents the number of students in the second school, what represents the number of students in the first school? in the two schools? What two expressions are there for the number of students in both schools?

34. Find two consecutive integers whose sum is 29.

35. Find three consecutive integers whose sum is 21.

36. The sum of two consecutive odd numbers is 72. What are the numbers?

37. The sum of two numbers is 13 and three times the smaller exceeds the greater by 3. What are the numbers?

38. The American Baseball League was organized in 1900. During the first 13 years, the Chicago, Detroit, and Boston teams won the pennant the same number of times, and the Philadelphia team won it once oftener than the Boston team. How many times did each win the pennant?

39. A farm of 160 acres has three times as many acres in pasture as in oats, twice as many acres in meadow as in oats, and 10 acres more in corn than in oats. Ten acres are in other crops. How many acres are there in each of the crops named?

40. The perimeter of a rectangle is 340 feet. The length is 10 feet less than twice the width. What is the length?

41. In the 63d Congress Iowa and California had the same number of representatives and Illinois had five more than twice as many as California. The three States together had 49 representatives. How many had each?

4-11.

REVIEW EXERCISES

40. **1.** What number exceeds a by 5?

2. What number is 12 less than the square of x?

3. What number is k greater than twice m?

4. What number is 10 less than the sum of p and q?

5. What is the difference between an exponent and a power? Define and give an example of each.

6. From the sum of 12, -4, 15, and -27 subtract the product of -3 and 13.

7. Solve: $4x + 12 - 3x = 6x - 4 - x$.

8. In a certain year the railroad mileage of Pennsylvania was 645 miles less than that of Illinois. The two states together had approximately 23,057 miles. What was the railroad mileage of each state?

9. Solve: $17a + 14 + 13a - 10 = (20 - 13)a - 19$.

2y+5 2y+5

10. The formula for finding the volume of a circular cylinder is

$$V = \pi r^2 h,$$

where V is the volume, r the radius of the base, and h the height of the cylinder. π is about $\frac{22}{7}$. Find the volumes of the following cylinders:

(a) $r = 4$ in., $h = 3$ in.
(b) $r = 6$ ft., $h = 10$ ft.
(c) $r = 10$ cm., $h = 24$ cm.

The amount (a) of a certain sum of money (p) bearing interest at a certain rate per cent (r) for a given time (t) (in years) may be found from the formula,

$$a = p\,(1 + rt)$$

11. Find the amount of $400 for 3 years at 6%. Here $r = \frac{6}{100}$.

12. Find the amount of $750 for 2 years at 6%.

13. Find the amount of $250 for 1 year 6 months at 5%.

14. Find the amount of $1000 for 3 years 8 months at $4\frac{1}{2}$%.

(9) **15.** The sum of three numbers is 58. The second is 3 times the first and the third is 2 more than the first and second together. Find the numbers.

16. If $a = 1$ and $b = -2$, find the value of

$$(a + b)^2 - 2a\,(a - b) + (a + b)(a - b)$$

17. If $m = -5$, $n = 3$, and $x = -8$, find the value of $m^2x - 7n^3 + 11m + x^2$.

18. Is the product of an even number of negative factors positive or negative? Is the product of an odd number of negative factors positive or negative?

19. If several numbers, some positive and some negative, are multiplied together, on what does the sign of the product depend?

20. What is the sign of an even power of a negative number? of an odd power of a negative number?

21. $\frac{3}{8}\cdot 32\cdot(-5)\cdot(-\frac{1}{2})\cdot(-\frac{1}{4})\cdot 3 = ?$

22. Find the value of the following when $x = -2$, $y = -3$, $m = \frac{3}{2}$, and $n = -\frac{2}{3}$:

(a) $(x^3 - y^3) + (x - y)$

(b) $4mn(m^2 - n^2)$

23. The sum of two consecutive even numbers is 86. What are the numbers?

24. B is 4 times as old as A. If A is x years old, what is B's age? What was the age of each five years ago? What will be the age of each in 10 years?

25. A has c dollars and B has m dollars. If A gives B x dollars, how many dollars will each have?

26. A has twice as much money as B. If A gives B $5, they will both have the same amount. How much has each?

SUMMARY

SECTION

37. Axioms.
38. Equations.
39. Solution of equations containing one unknown.
40. Review exercises.

CHAPTER VI

SUBTRACTION

41. Subtraction of Monomials. We saw (section 26) that subtracting a number is equivalent to adding the same number with its sign changed.

Thus, $12 - (+5) = 12 + (-5) = 12 - 5 = 7$

and $12 - (-5) = 12 + 5 = 17$

ILLUSTRATIVE EXERCISES

1. Subtract $15a$ from $12a$.

Solution:

$$12a - 15a = 12a + (-15a) = [12 + (-15)]a = -3a$$

2. Subtract $-3a$ from $7a$.

Solution:

$$7a - (-3a) = 7a + 3a = (7 + 3)a = 10a$$

3. From -24 subtract 64.

Solution:

$$\begin{aligned} -24 &= -3 \cdot 8 \\ 64 &= 8 \cdot 8 \\ \hline -11 \cdot 8 &= -88 \end{aligned}$$

Perform mentally the following subtractions:

	1.	**2.**	**3.**	**4.**	**5.**	**6.**
From	$35 = 7 \cdot 5$	-27	44	-72	-49	$10 \cdot 12$
Take	$25 = 5 \cdot 5$	-45	± 88	-54	-63	$-7 \cdot 12$

	7.	**8.**	**9.**	**10.**	**11.**	**12.**
From	$33a$	$14a$	$2b$	$-9ac$	$15k^2$	$12bt$
Take	$-18a$	$-3a$	$-4b$	$-2ac$	$-4k^2$	$-7bt$
	$+15a$	$17a$				

	13.	**14.**	**15.**	**16.**	**17.**	**18.**
From	$-9ac$	$7ak^2$	$4c^2s$	$24a^2b$	$-33cd^2$	$56m^2n^2$
Take	$3ac$	$3ak^2$	$10c^2s$	$-14a^2b$	$-5cd^2$	$-9m^2n^2$

	19.	**20.**	**21.**	**22.**	**23.**
From	$-17bcm$	$66abc^2$	$-56x^3y^3z^3$	$\frac{3}{4}ab$	$\frac{1}{2}c$
Take	$56bcm$	$-33abc^2$	$-18x^3y^3z^3$	$\frac{1}{2}ab$	$-\frac{1}{2}c$
			8		

	24.	**25.**	**26.**	**27.**	**28.**
From	$7 \cdot 8$	$-3 \cdot 19$	$-12 \cdot 29$	$17x^2z$	$\frac{2}{3}m^2$
Take	$-4 \cdot 8$	$7 \cdot 19$	$-7 \cdot 29$	$-7x^2z$	$\frac{1}{6}m^2$
	11.8			24	

	29.	**30.**	**31.**	**32.**
From	$-\frac{2}{3}x^2y$	$6(a+b)$	$-12(x-y)$	$-3(m^2+n^2)$
Take	$-\frac{1}{6}x^2y$	$-9(a+b)$	$17(x-y)$	$-23(m^2+n^2)$
				20

If the coefficients of the common factor are literal expressions, the subtraction can be indicated only.

Thus, $9ax^2 - 4cx^2 = (9a - 4c)x^2$
and $9ax^2 - (-4cx^2) = (9ax^2 + 4cx^2) = (9a + 4c)x^2$

33. From $3ab$ subtract $7ab$. $-4ab$
34. From $7pq$ subtract $-7mq$.
35. From $3cp^2$ subtract $-2mp^2$.
36. From $8x(a+b)$ subtract $7y(a+b)$
37. From $-9ab$ take $-4ac$
38. From $12a^2d$ take $-5a^2c$
39. From $2a(m-n)$ subtract $-b(m-n)$
40. Take $a^2(c+d)$ from $b^2(c+d)$
41. From $3b(x-1)$ subtract $(x-1)$
42. From $-2ab(h+k)$ subtract $-cd(h+k)$

42. Subtraction of Polynomials.

ILLUSTRATIVE EXERCISE

From $3ax - 2by + 7z^2$ subtract $4by - 7ax - 8z^2$

In this exercise each of the numbers $4by$, $-7ax$, and $-8z^2$, is to be subtracted from $3ax - 2by + 7z^2$. We have found (section 26) that subtracting $4by$ is equivalent to adding $-4by$, subtracting $-7ax$ is equivalent to adding $+7ax$, and subtracting $-8z^2$ is equivalent to adding $+8z^2$. The problem, then, may be made one of addition, namely, to add $3ax - 2by + 7z^2$ and $-4by + 7ax + 8z^2$. The student will notice that the last expression is the expression to be subtracted in the original problem with the signs of its terms changed. We have then the following rule for subtraction:

Change the sign of each term in the subtrahend and proceed as in addition.

NOTE. The instruction in section 26 should be kept in mind and the changes in signs made mentally. The changes should not be made on board or paper.

Solution:

The above exercise may be conveniently worked as follows:

$$\begin{array}{r@{\qquad}r} 3ax - 2by + 7z^2 & 8 \\ -7ax + 4by - 8z^2 & -11 \\ \hline 10ax - 6by + 15z^2 & 19 \end{array}$$

The $10ax$ is obtained by adding $+7ax$ to $3ax$, the $-6by$ by adding $-4by$ to $-2by$, and the $+15z^2$ by adding $+8z^2$ to $+7z^2$. Check by letting each letter equal 1 and writing the results at the side as above.

EXERCISES

1. From $7a - 3y$ subtract $3a - 2y$. Check for $a = 1$ and $y = 1$.

2. From $10a - 6b$ subtract $7a + 2b$. Check for $a = 1$ and $b = 1$.

3. From $3x - 4y$ subtract $5x - 6y$.

4. From $8a - 7b + 6c$ subtract $-6a + 7b - 6c$.

5. From $-4x - 5y + 12z$ subtract $5x + 12y + 7z$.

6. From $7ax - 2cy - 4m^2$ subtract $3cy - 10ax + 7m^2$.

7. Take $-2m^2 - 3m^3 + m$ from $m^3 + m^2 + m$.

8. Take $5a^3 + 17a^2b - ab^3 + 9b^4$ from $13a^3 - 8a^2b + 4ab^3 - b^4$.

9. From the sum of $3mn + bc$, $-4bc + xy$, and $2ad - 2xy$, take $2mn + 2bc + xy + ad$.

10. From $7x^3 - 4x^2 + 3x + 1$ take $x^3 - 7x^2 + 3x - 4$.

11. From $6(a + b) + 3(c + d)$ take $-3(a + b) - 7(c + d)$.

12. From $8ax + 4by$ subtract $2ax - 3by$.

13. What number added to $3x - 2y + 5z$ will give $5x - 3y - 6z$?

14. What number added to $4a - 2b + 3c$ will give $7a + b - 2c$?

15. What number added to $3a - 6m + 4x$ will give $5a$?

16. From $12x^3 + 17x^2z + 7ab^2 + 20b^3$ subtract $5x^3 - ab^2 + 17x^2z - 2b^3$.

17. From $7x^4 - 4x^2 + 3x + 1$ take $4x^3 - 2x^2 - 2x - 14$.

18. From the sum of $x^3 - 3x^2y + 3xy^2 - y^3$ and $x^3 + 3x^2y + 3xy^2 + y^3$ take $3x^3 - 3y^3$.

19. From $\frac{1}{2}x + \frac{1}{3}y$ take $\frac{1}{4}x - \frac{1}{4}y$.

20. Subtract $\frac{3}{4}a^2b - \frac{2}{3}ab^2$ from $2a^2b - ab^2$.

REVIEW EXERCISES

43. 1. From the sum of $3ax^2 + 2ay + 3cd$ and $4cd - 2ax^2 - 8ay$, take $3ay - 2cd + 3ax^2$. Check result.

2. Subtract $xy - 3mnx^2 + 12kl$ from $2kl - 4mnx^2 - 11xy$, and check the result by letting each letter equal 2.

3. What number added to $4x - 3y$ gives $11x - y$?

4. The length of a rectangle is 13 feet more than twice the width, and the perimeter is 68 feet. Find the length and width.

5. From the sum of $3ks + 3c^3 - 4y^2$ and $4c^3 + 2y^2 - 5ks$ take $2c^3 - 3ks - 3y^2$. Check by putting each letter equal to 1.

6. What number exceeds the product of a and b by x? What number is x less than the product of a and b?

7. Solve: $8x + 10 = 2x + 4$.

8. A man can row 6 miles an hour in still water. How fast can he row up a stream flowing 2 miles an hour? down the stream?

9. A man can row m miles an hour in still water. How fast can he row up a stream flowing x miles an hour? down the stream?

10. Solve: $7x - 3x - 11 = 5x - 8 - 2x$.

11. Find the average mean temperature at a certain place for January if the mean for each day was as follows:

1st	− 10°	9th	+ 20°	17th	− 8°	25th	− 32°
2d	− 2°	10th	+ 12°	18th	− 13°	26th	− 29°
3d	+ 1°	11th	+ 5°	19th	− 18°	27th	− 20°
4th	− 6°	12th	+ 4°	20th	− 24°	28th	− 12°
5th	− 10°	13th	+ 8°	21st	− 28°	29th	+ 6°
6th	− 8°	14th	− 13°	22d	− 25°	30th	+ 15°
7th	+ 3°	15th	− 19°	23d	− 27°	31st	+ 25°
8th	+ 4°	16th	− 13°	24th	− 22°		

12. What number subtracted from $3x^2 - 12x + 14$ gives $3x + x^2 - 3$?

13. If three times a number be subtracted from the number, the remainder is 6. What is the number?

14. Express in algebraic symbols: The square of the sum of two numbers is equal to the square of the first plus twice the product of the two numbers plus the square of the second.

15. Find the value of $3m^3 - 10m^2 + 15m - 25$ when $m = -2$; when $m = -\frac{1}{3}$.

16. A boy has a dimes and b nickels. How many cents has he?

17. If $a = -4$, $b = -2$, and $m = 4$, find the value of

(a) $5ab - 2m + 12bm$

(b) $\sqrt{2abm} + \dfrac{am}{b} - (a - b)\,2m$

18. Subtract the sum of $3ax + 7by$, $2mn - 6ax$, and $10by - 3mn$, from the sum of $2mn - 3ax + 11by$ and $6ax - 9by + 3mn$.

19. $s = \frac{1}{2}gt^2$ is a formula used for finding the distance an object dropped from a height will fall in a given time. s represents the distance, t the time in seconds, and g is equal to about 32 feet. By substituting 32 for g and the time given for t, find how many feet a body dropped from a height will fall in

(a)	5 seconds	(c)	4 seconds	(e)	$2\frac{1}{2}$ seconds
(b)	10 seconds	(d)	1 second	(f)	1 minute

SUMMARY

SECTION

41. Subtraction of monomials.

42. Subtraction of polynomials.

43. Review exercises.

CHAPTER VII

SYMBOLS OF GROUPING

44. Removal of Parentheses.

ILLUSTRATIVE EXERCISES

1. $16 + (8 - 3)$ means that the number in the parenthesis is to be added to 16. The numbers in the parenthesis may be combined and the resulting number added to 16, or 8 and -3 may be added separately.

Thus, $16 + (8 - 3) = 16 + 5 = 21$

or, $16 + (8 - 3) = 16 + 8 + (-3) = 16 + 8 - 3 = 21$

2. $16 - (8 - 3)$ means that the number $(8 - 3)$ is to be subtracted from 16. The rule for subtraction gives

$$16 - (8 - 3) = 16 - 5 = 11,$$

or $16 - (8 - 3) = 16 + (-8) + (+3) = 16 - 8 + 3 = 11$

3. $3a - 7b + (5a - 3b)$ means that $5a - 3b$ is to be added to $3a - 7b$. The sum is $8a - 10b$. The expression $3a - 7b + (5a - 3b)$ may be written $3a - 7b + 5a - 3b$.

4. $3a - 7b - (5a - 3b)$ means that $5a - 3b$ is to be subtracted from $3a - 7b$. The rule for subtraction gives

$$\begin{aligned} 3a - 7b - (5a - 3b) &= 3a - 7b + (-5a) + (+3b) \\ &= 3a - 7b - 5a + 3b = -2a - 4b \end{aligned}$$

The following rule for removing parentheses results:

A parenthesis preceded by a plus sign may be removed and the terms within written with signs unchanged.

A parenthesis preceded by a minus sign may be removed if the sign of each term within it is changed.

(It should be noted that the sign of the parenthesis is dropped with the parenthesis.)

When a parenthesis is included within a parenthesis or other symbol of grouping, it is best to remove *one* parenthesis at a time. The student is usually advised to remove the inner parenthesis first, but this is not essential.

ILLUSTRATIVE EXERCISES

1. Remove the symbols of grouping and combine the terms of $3a - [\,6b + 2a - (\,3a - b\,)\,]$

Solution:

Removing the parenthesis and collecting terms:

$$\begin{aligned} 3a - [\,6b + 2a - (\,3a - b\,)\,] &= 3a - [\,6b + 2a - 3a + b\,] \\ &= 3a - [\,7b - a\,] \end{aligned}$$

Removing the brackets and collecting terms:

$$\begin{aligned} 3a - [\,7b - a\,] &= 3a - 7b + a \\ &= 4a - 7b \end{aligned}$$

2. Find the numerical value of $16 + \{5 - [8 + (6 - 3)]\}$

Solution:

Collecting terms and removing the symbols of grouping beginning with the parenthesis:

$$\begin{aligned} 16 + \{\,5 - [\,8 + (\,6 - 3\,)\,]\,\} &= 16 + \{\,5 - [\,8 + 3\,]\,\} \\ &= 16 + \{\,5 - 11\,\} \\ &= 16 + \{\,-6\,\} \\ &= 16 - 6 \\ &= 10 \end{aligned}$$

3. Find the numerical value of $8 + [\,2 - (\,3 - \overline{7 - 2} + 3\,) - 4\,]$

Solution:

Collecting terms and removing the symbols of grouping beginning with the vinculum:

$$\begin{aligned} 8 + [2 - (3 - \overline{7 - 2} + 3) - 4] &= 8 + [2 - (3 - 5 + 3) - 4] \\ &= 8 + [2 - (1) - 4] \\ &= 8 + [-3] \\ &= 5 \end{aligned}$$

EXERCISES

In the following problems remove the symbols of grouping and combine when possible. Solve the simpler exercises mentally.

1. $15 + (8 - 2)$ - 21
2. $12 - (3 + 6)$
3. $3x - (2a + 3b)$
4. $4c - (2x - 6)$
5. $6z - (-3 + 7y)$
6. $x + (3x - 5)$
7. $4 + (4 - 2) + 3 - (6 - 2)$
8. $8 - (-3 + 6)$
9. $7 - 2 - (3 - 1) - (2 - 3)$
10. $(9 - 4) - [5 - (3 - 4) - 3]$
11. $12x - (3x + 4)$
12. $7a - (2a - 4)$
13. $2k - (3k - 4) + (4k - 5)$
14. $(x + 5) - (y + 7)$
15. $8z - (3z + 2) - (2z - 7)$
16. $13x - (17x - 10) - (14x - 13)$
17. $-7x + 3y - 14z - (24y - 15z + 12x)$
18. $3ab - (-8x - 7y + 12) + (10ab - 6)$
19. $3 - [t - 6t - (t - 8t) - (7t + 3)]$
20. $12ab - (3ab - 4cx + 12a)$
21. $x^3 - (5x^2 - \overline{8x - 7})$
22. $x^2 + 1 - \{x^2 - (2x - 3) + 2\}$
23. $3 + (7w - \overline{4w + 9 - 2s}) + (7w + \overline{5s - 8})$
24. $24rs - 7s + 2r - (15rs - 12s - 13r) - (12r - s)$

Solve the following equations:

25. $12 + x - (5x + 10) = -2$
26. $5x - (6 - 2x) + x = 2x + 12$
27. $7x - [3x + (5 - x)] = 15$

28. A and B are 32 miles apart and travel toward each other, A going 2 miles an hour faster than B. They meet in 4 hours. How fast does each travel?

29. The tens' digit of a number is 2 greater than the ones' digit. The number is 2 more than 10 times the tens' digit. What is the number?

45. Insertion of Parentheses.

Since a parenthesis preceded by a plus sign may be removed without altering the value of the expression, it is evident that any group of terms may be enclosed in a parenthesis preceded by a plus sign without changing the value of the expression.

Thus, $3 + 7 - 5 + 6 - 4 = 3 + (7 - 5) + (6 - 4)$
and $7 - 3 + 4 - 5 = 7 + (-3 + 4 - 5)$

Since a parenthesis preceded by a minus sign may be removed provided the sign of each term within the parenthesis is changed, it is evident that a group of terms may be included in a parenthesis preceded by a minus sign provided the sign of each term placed in the parenthesis is changed.

Thus, $8 + 6 - 4 - 7 + 8 = 8 - (-6 + 4) - (7 - 8)$
and $15 - 4 + 7 - 3 + 5 = 15 - 4 - (-7 + 3 - 5)$

EXERCISES

In the following expressions enclose in parentheses terms containing a common factor:

1. $4x + 5x + 6y + 13y$

2. $2aby - 4cx + 3aby + 2cx$

3. $3ab - 5ax - 5by + 7xy$

4. $-7a + 10b + 21a + 20$

5. $3abx - 5rt^5 - 3y - 5a$

6. $am - bm - 3x - 4ax - 7bx$

7. $4am - 6ax - 5by - 7ry$

8. $6mx^2 - 2aby^2 - mx^2 - ax^2 + 2by^2$

9. $4y^2 + 2a^2x^2 - 2 - 12y^2$

10. $ax^2 - b^2y^2 - cz^2 + 12y^2 - 7x^2 - z^2$

Place the last three terms of each of the following expressions in a parenthesis preceded by the minus sign:

11. $6a - (4b + 4c - 2b)$

12. $2x^2 + 3x - 3y + 3$

13. $a^2 - b^2 + 2bc - c^2$

14. $9 - 4x^2 - 12xy - 9y^2$

15. $3x - 7xy + 4y^2 - 3y + 7$

16. $2z + 3y - 3x + 8z - 7$

17. $x^2 + 2xy + y^2 - 2mn + m^2 + n^2$

18. $16a^2 + 4x^2 + 4x - 1$

SUMMARY

SECTION

44. Removal of parentheses. Problems.
45. Insertion of parentheses. Problems.

24 Problems

$(2aby + 3aby) - (+4 - 4 - 2c$

CHAPTER VIII

CONDITIONAL EQUATIONS AND IDENTITIES

46. Definitions. In section 38 an equation was said to express the equality of two numbers or expressions. That is, the two members of an equation represent the same number. Such equations as

$$2 + 4 = 6 \qquad (1)$$

$$\text{and } 5 \cdot 4 = 20 \qquad (2)$$

are always true. Also such an equation as

$$4x + 5x = 9x \qquad (3)$$

is true no matter what number is represented by x.

On the other hand, the equations,

$$x + 2 = 7 \qquad (4)$$

$$\text{and } 5x = 30 \qquad (5)$$

are true for only one value of x in each case. Thus equation 4 is true only when $x = 5$ and equation 5 is true only when $x = 6$.

Equations which are always true, like 1, 2, and 3, are called *identical equations* or *identities.* The symbol for identity, $\equiv$, is often used to state the fact that two numbers are identically equal. For example,

$$4x + 5x \equiv 9x$$

Equations which are true for one or more values, but not for all values of the unknown (or unknowns) involved are called *conditional equations.* Equations 4 and 5 are conditional equations.

The equations thus far solved are conditional equations, and equations are understood to be conditional unless otherwise stated.

47. Solution of Equations. To solve an equation involving one unknown is to find a value (or values) of the unknown, which, when substituted for the unknown in the original equation, makes that equation an identity. Such a value of the unknown is called a *root* of the equation.

If, in equation 4, 5 is substituted for x, the equation reduces to

$$5 + 2 = 7$$

an identity, and 5 is said to *satisfy* the equation. What is 5 called?

ILLUSTRATIVE EXERCISE

Solve: $4x - (3x + 3) + 4 = 5x + (5 - 6x)$

Solution:

Removing parentheses,

$$4x - 3x - 3 + 4 = 5x + 5 - 6x$$

Combining like terms in each member,

$$x + 1 = -x + 5$$

Adding x to each member (Axiom II),

$$2x + 1 = 5$$

Subtracting 1 from each member (Axiom III),

$$2x = 4$$

Dividing each member by 2 (Axiom V),

$$x = 2$$

Check: Substituting 2 for x in the original equation,

$$8 - (6 + 3) + 4 = 10 + (5 - 12)$$
$$12 - 9 = 10 - 7$$
$$3 = 3$$

Steps 3 and 4 could have been combined by adding $x - 1$ to each member.

B

EXERCISES

Solve the following equations. Check result in each case by substituting in the original equation the value obtained for the unknown. Solve the simpler equations mentally.

1. $x - 5 = 8$
2. $x + 7 = 3$
3. $4x + 2 = 10$
4. $\frac{m}{3} = 4$ $m = 12$
5. $2x - 3 = x + 1$
6. $3x - (x + 2) = 14$
7. $10m + (4 - 3m) = 2m + 7$
8. $2x - \frac{3}{2}x = -4$
9. $6x - (5 - 2x) = 35$
10. $7y - (11 - 4y) = 9 + (3y - 8)$
11. $6x - (3 + x) = 2x + 12$
12. $4x + (3x - 5) = 16$
13. $3s + 2s - s = s + 8$
14. $4x - 7 = 3 + 2x$
15. $4y + (y - 5) = 35$
16. $3s - (s + 6) = -8$
17. $12x + 4 = 8x - (x - 29)$
18. $22 - (3x - 7) = 7x - 1$
19. $7x - 5 - (8x + 6) = -9$
20. $3m + 2 + (4m - 3) = \frac{5}{2}$
21. $15 - (n + 24) = 4n + 18$
22. $11 - 5n + 7 - 3n = 3 - 11n + 23$
23. $4x - 2 = \frac{11}{3}x - 6$
24. $1.5a + 10 = .9a + 16$
25. $2z - (8 - 7z) = 6z - (8 - 4z) + 42 - 5z$
26. $\frac{5}{3}w - \frac{1}{3} = w + \frac{3}{2}$
27. $\frac{8}{5}x - 1 = \frac{3}{5}x + \frac{4}{5}$
28. $6x - .4 - (12x - 2.5) = 1.5$
29. $3c - 12 + (4c - 2) = 13 - (5 - 3c)$
30. $6x + (24 - 3x) - (15x + 2) = 98$

31. A's money is \$70 more than twice B's and they both have \$280. How much has each?

Suggestion: Let $x =$ the number of dollars B has. What represents the number of dollars A has? What two expressions are there for the number of dollars they both have?

32. A horse cost \$10 more than 3 times as much as a cow, and both cost \$250. What was the cost of each?

33. Divide 120 into three parts such that the second part is 15 more than the first and the third part exceeds 3 times the first by 5.

34. One-half of a number increased by 50 is 10 less than twice the number. What is the number?

35. The sum of three consecutive numbers is 66. What are the numbers?

36. The sum of three consecutive integers is 41 more than the largest. What are the numbers?

37. The sum of three numbers is 53. The second is twice the first and the third is 5 more than 3 times the first. Find the numbers.

38. A man bought a hat, a pair of shoes, and an overcoat for \$33. The hat cost \$2 less than the shoes and the overcoat cost \$1 more than 8 times as much as the hat. What was the cost of each?

39. The sum of three numbers is 51. The second number is 3 times the first and the third is 3 more than the sum of the other two. Find the numbers.

40. The length of a lot is 30 feet more than twice the width and the perimeter is 420 feet. What are the dimensions of the lot?

41. The Colorado river is as long as the Ohio and the Connecticut combined and the Ohio is 130 miles longer than twice the length of the Connecticut. The combined length of the three rivers is 2720 miles. What is the length of each?

42. A cubic foot of ivory weighs approximately 47 pounds less than a cubic foot of marble and 53 pounds less than a cubic foot of glass. Three cubic feet of ivory weigh 15 pounds more than a cubic foot of glass and a cubic foot of marble. Find the weight of a cubic foot of each.

REVIEW EXERCISES

48. **1.** Simplify $4 - 6 - [14 - (8 - 10 - 2 + 4)]$

2. If $a = 2$, $b = 3$, $x = 1$, and $y = 2$, find the value of

(a) $3ax - 5by - (7by + 3ax - 5ax - 3by)$

(b) $11a - (7b - 3a) + 5b - [3a - (4b + 5a)]$

3. Enclose the last three terms of each of the following in a parenthesis preceded by $-$:

(a) $8 - 3y + ay + 6aby$

(b) $23a + 15 + 4ax - 11x + 17ax$

4. Remove the parentheses and simplify:

(a) $3cd - (c^2 - 3cd + 4k - 4k - 5cd + c^2)$

(b) $2m + 3n - (4n - p + m) - (5m - 7n + 3m)$

5. Add $7abc - 3xyz + 11mn$, $-4xyz + 121mn$, $3xyz - 41mn + 5abc$, and $8abc - 21mn$.

6. $(3a + 4b) + (5b + 11c) - \{(7a - 4b) - (5a - 6b)\} = ?$

7. From the sum of $5x^2 + 4ay + 3y^2$ and $2x^2 - 5ay + 6y^2$ take $3x^2 - 8ay + 8y^2$.

8. Solve $7x - 12 + 3 = 5x + 16 - 5$.

9. Solve $7x - (14 - 4x) = 7 - (3x - 8)$.

10. State in words what $a - x$ represents; $a^2 - x^2$.

11. A has \$5 more than 6 times as much money as B, and together they have \$145. How much has each?

12. The sum of two numbers is 71 and the greater exceeds 16 times the smaller by 3. What are the numbers?

13. What number added to $5a^3 + 6a^2b - 7ab^2 + 11b^3$ gives $10b^3 - 3a^2b + 5ab^2 - 12a^3$?

14. In a certain year Columbia University had 99 less than 4 times as many students as Princeton and 325 more than the University of Wisconsin. The total number of students in the three universities was 13,364. How many were there in each?

15. In 1910 France had 7 more modern battleships than Japan, the United States 4 more than twice as many as Japan, Germany 1 less than the United States, and Great Britain 1 more than Germany and France together. These 5 nations had 133 modern battleships. How many had each?

16. Divide 41 into 3 parts such that the first is 1 more than 4 times the second and the third is 1 more than 8 times the second.

17. Add $ax^2 + by^2$ and $ax^2 - dy^2$.

18. From $mx^3 - ny^3$ subtract $3x^3 + 4y^3$.

19. Solve: $2x - (22 - 7x - 14) = 6x - (8 - 4x) + (42 - 5x)$.

20. A garden is 10 feet more than 4 times as long as wide and its perimeter is 620 feet. What are the length and the width?

21. If $x = -2$, and $y = -3$, find the value of

$$3xy - \{4x^2 - y^2 - (2xy + 3x^2 - 5y^2)\}$$

22. A boy bought x oranges at 3 cents each and y bananas at 2 cents each. What did he pay for all?

23. A had m dollars and B n dollars. A gave B s dollars. How much did each then have?

24. The ones' digit of a number is one more than the tens' digit. If x represents the tens' digit, what represents the ones' digit? What represents the number? the sum of its digits?

25. The ones' digit of a number is 1 more than the tens' digit. The number is 3 more than 4 times the sum of its digits. What is the number?

26. The ones' digit of a number is 2 greater than the tens' digit. The number is 7 more than 6 times the tens' digit. What is the number?

SUMMARY

SECTION

46. Conditional equations and identities defined.
47. Solution, or root, of an equation. Problems.
48. Review exercises.

21, 24, 25, 26,

Given up, but not forgotten

[redacted]

CHAPTER IX

MULTIPLICATION

49. Law of Order. From arithmetic we know that changing the order in which the factors are multiplied together does not affect the product.

For example, $4 \cdot 5 = 5 \cdot 4$

$$7 \cdot 6 \cdot 3 = 7 \cdot 3 \cdot 6 = 3 \cdot 7 \cdot 6 = 6 \cdot 3 \cdot 7 = 3 \cdot 6 \cdot 7 = 6 \cdot 7 \cdot 3$$

This principle is called the **commutative law of multiplication.** We assume this law to hold in algebra also. Thus,

$$a \cdot b \cdot c = a \cdot c \cdot b = b \cdot c \cdot a$$

Law of Grouping. We also assume that the factors of a product may be grouped in any order without affecting the product. This is called the **associative law of multiplication.**

Thus, $4 \cdot 6 \cdot 10 = 4 \cdot (6 \cdot 10) = (4 \cdot 6) \cdot 10 = (4 \cdot 10) \cdot 6$

$$a \cdot b \cdot c = a \cdot (b \cdot c) = (a \cdot b) \cdot c = (a \cdot c) \cdot b$$

50. Multiplication of monomials. The product of two numbers is the product of the factors of the numbers. For example,

$$24 \cdot 35 = (2 \cdot 2 \cdot 2 \cdot 3) \cdot (5 \cdot 7) = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 7 = 840$$

$$3ax \cdot 7by = (3 \cdot a \cdot x) \cdot (7 \cdot b \cdot y) = 3 \cdot 7 \cdot a \cdot b \cdot x \cdot y = 21abxy$$

To find the product of two or more monomials:

Find the product of the literal factors and, for the coefficient of this product, the product of the numerical coefficients.

If a letter occurs more than once as a factor, an exponent indicates how many times it is so used. For example,

$$a^2 \cdot a^3 = (a \cdot a) \cdot (a \cdot a \cdot a) = a^5$$

$$3ax^2 \cdot 4ax = 12a \cdot a \cdot x^2 \cdot x = 12a^2x^3$$

Since the exponent tells how many times a number is taken as a factor,

The exponent of any number in the product is equal to the sum of the exponents of that number in the factors.

Stated in a formula, this is,

$$\mathbf{a^m \times a^n = a^{m+n}}$$

Caution. When no exponent is written, the exponent 1 is understood.

EXERCISES

1. $2 \cdot 5 = ?$ $8 \cdot 27 = ?$ Multiply $5 \cdot 6$ by 4. Multiply $3 \cdot 7 \cdot 4$ by 2.

2. Multiply the following products by 5:

$3 \cdot 5$ $2 \cdot 5^2$ $8 \cdot 3 \cdot 2$ $4 \cdot 5 \cdot 9$ $7 \cdot 3 \cdot 8$ $3 \cdot 6 \cdot 5 \cdot 2$

3. Multiply the following:

x^2 by x^3
a^2x^3 by a^3x^2
$3a^2x$ by $2a^3x$
$7b^2y$ by $2by^3$
$6m^2by$ by $3m^4$
$8abx^2$ by $2a^3$
$9y^2z$ by $2z^3x$
$8a^5b^2x$ by $2b^4c^2x$

4. Find the products of the following:

$17a^3b$ and $11ab^2$
$7m^2np$ and $13mp^2$
$18r^2s^3t^4$ and $6rs^2t^2$
$12x^5y^4z^3$ and $11x^2y^3z$
$15u^2v$ and $4uv^2$
$6a(b + c)$ and $3x(b + c)^2$
$7mn(x + y)$ and $2m^2n^2(x + y)^3$
$6k^3(a + x)$ and $3k^2(a + x)$

Section 28 gives the law of signs for multiplication. What is this law?

What is the sign of the product of an odd number of negative factors? of an even number of negative factors?

ILLUSTRATIVE EXERCISES

1. $(-4)\cdot 3\cdot(-5)\cdot(-2)\cdot 6\cdot 2 = ?$ – 1440

Notice that there is an odd number of negative factors. What is the sign of the product?

2. $7\cdot(-2)\cdot 3\cdot(-4)\cdot 2 = ?$ 336

Notice that there is an even number of negative factors. What is the sign of the product?

EXERCISES

Find the following products mentally:

1. $-3\cdot 5$ – 15
2. $6\cdot(-5)$ – 30
3. $3\cdot(-7)\cdot 5$ – 105
4. $4\cdot 5\cdot(-3)$ – 60
5. $(-2)\cdot 3\cdot(-5)$ 30
6. $4\cdot 5\cdot 3\cdot(-6)$ – 360
7. $(-2)\cdot(-3)\cdot(-5)$ – 30
8. $3a^2x\cdot 5x^3$ [illegible]
9. $-2b^2y^2\cdot 3b^2y$
10. $7c^2xy\cdot(-2ax^2)$
11. $-3x^2z^3\cdot(-4ax^2)$
12. $5a^2bc\cdot(-2ac^3)$
13. $a^2x^3\cdot 3a^5x^2$
14. $2x^3y\cdot 4xy^2$
15. $5z^2y\cdot(-2zy^2)$
16. $4x^2z^4\cdot(-6xy^2z^2)$
17. $3^2z\cdot 3^3z^2x$
18. $3ac^2\cdot(-6a^2b)\cdot(-5abc)$
19. $(-3mnp)\cdot(-4m^4n^2)\cdot(-mq^2)$
20. $(-6x^3yz)\cdot(2x^2z^2)\cdot(3y^4z^5)$
21. $(-3m^2n^2p)\cdot(-2mn^2)\cdot(4m^2p^4)$
22. $(-2ab^2)\cdot(\frac{1}{2}a^2b)\cdot(-4ab^3)$
23. $(-\frac{2}{3}xy)\cdot(-3x^3y^2)\cdot(-\frac{3}{4}x^2)$
24. $-5x\cdot(a-b)\cdot 3x^2(a-b)^2\cdot\frac{1}{2}(a-b)$
25. $-\frac{2}{5}m(x+y)\cdot(-\frac{2}{3}n(x+y)^2)\cdot(-\frac{3}{2}mn(x+y))$

51. Multiplication of a Polynomial by a Monomial.

1. Multiply $3 + 5$ by 2.

This may be done by first adding 3 and 5 and then multiplying the sum by 2, or by multiplying 3 and 5 separately by 2 and adding the products so obtained. Thus,

$$3 + 5 = 8 \qquad 8 \cdot 2 = 16$$

Or,

$$\begin{array}{r} 3 + 5 \\ 2 \\ \hline 6 + 10 = 16 \end{array}$$

2. Multiply $2a + 3b$ by 4.

$$\begin{array}{r} 2a + 3b \\ 4 \\ \hline 8a + 12b \end{array}$$

To multiply a polynomial by a monomial:

Multiply each term of the polynomial by the monomial and connect the products by their proper signs.

EXERCISES

Multiply mentally:

1. $7 + 11$ by 8
2. $13 - 2$ by 4
3. $4ax^2 + 3by^2$ by 2
4. $6a - 5y$ by 3
5. $3x + 4y$ by -5
6. $3x - 2k$ by $3a$
7. $5x - 3$ by $2x^2$
8. $-4a + 5b$ by $-2a$
9. $-3ax - 6by$ by $-3x$
10. $8xy^2 - 4y^2z$ by $-2x^2y$
11. $a^2 + 2ab + b^2$ by $-4ab$
12. $4a^2b - 3b^2z + 2b^2$ by $5bc$
13. $2m^2n + 12n^2p^3 - 5pqr$ by $-4mpq$
14. $x^3 - x^2 + x - 1$ by $-x^2$
15. $4uv - 2u + 7$ by $3uv$

Algebra

16. $2m^2t - 7m^3 + 8tx^2$ by $-at$
17. $(a + b - c)x = ?$
18. $x(a + b) + x(a - b) = ?$
19. $(x^2 + x - 5)2x = ?$
20. $x(x^2 + 5) + 3x^2(x + 2) = ?$

Multiply:

21. $12m^2 - 8m + 16$ by $\frac{1}{2}$
22. $10p^2q - 25pq + 5q^2$ by $.4$
23. $4x^2y^2 - 6xy^2 + 8y$ by $1.5x$
24. $15m^3 - 21m^2n + 33mn^2$ by $\frac{2}{3}n$
25. $\frac{2}{3}my + \frac{1}{2}mx + \frac{1}{4}ny$ by $-12xy$

52. Multiplication of a Polynomial by a Polynomial. In arithmetic 634 is multiplied by 37 by multiplying 634 by 7 and then by 30 and adding the results.

ILLUSTRATIVE EXERCISES

1. Multiply 634 by 37.

Solution:

$$\begin{array}{r l}
634 & \\
\underline{37} & \\
4438 & = 634 \cdot 7 \\
\underline{19020} & = \underline{634 \cdot 30} \\
23458 & = 634 \cdot 37
\end{array}$$

To multiply by $a + b$, proceed in the same manner, that is, multiply by a and b separately and add the results thus found.

2. Multiply $a^2 + 2ab + b^2$ by $a + b$

Solution:

$$\begin{array}{l l}
a^2 + 2ab + b^2 & \\
\underline{a + b} & \\
a^3 + 2a^2b + ab^2 & = (a^2 + 2ab + b^2) \cdot a \\
\underline{\quad + a^2b + 2ab^2 + b^3} & = (a^2 + 2ab + b^2) \cdot b \\
a^3 + 3a^2b + 3ab^2 + b^3 & = (a^2 + 2ab + b^2) \cdot (a + b)
\end{array}$$

As the order of terms in addition does not affect the result, it is customary to find, as above, the product by the first term of the multiplier and then the product by the second term of the multiplier, etc. The products, $a^3 + 2a^2b + ab^2$ and $a^2b + 2ab^2 + b^3$, are called **partial products.**

To multiply one polynomial by another:

Multiply either polynomial by each term of the other and add the products thus obtained.

The partial products should be arranged so that like terms will be in the same column.

3. Multiply $x^2 - xy + y^2$ by $x + y$.

Solution:

$$\begin{array}{lllll}
\multicolumn{4}{l}{x^2 - xy + y^2} & \\
\multicolumn{4}{l}{x + y} & \\ \cline{1-4}
x^3 & - x^2y & + xy^2 & & \text{(Product by } x\text{)} \\
 & + x^2y & - xy^2 & + y^3 & \text{(Product by } y\text{)} \\ \cline{1-4}
x^3 & & & + y^3 &
\end{array}$$

Check by putting $x = 1$ and $y = 1$

$$\begin{aligned}
x^2 - xy + y^2 &= 1 \\
x + y &= 2 \\
x^3 + y^3 &= 2 \\
1 \cdot 2 &= 2
\end{aligned}$$

Since the product of the numbers obtained by substituting in multiplier and multiplicand equals the number obtained by substituting in the product, the work checks.

The exponents cannot be checked by giving to the letters the value of 1.

EXERCISES

Perform the following multiplications, checking when not certain that the result is correct:

1. $(x+1)(x+2)$
2. $(a+3)(a+2)$
3. $(x+3)(x+4)$
4. $(y-2)(y+7)$
5. $(a+b)(a-b)$
6. $(2y-3x)(x-y)$
7. $(3x+2)(x-2)$
8. $(x-3y)(x+3y)$
9. $(3y-2)(y+3)$
10. $(x^2+xy+y^2)(x-y)$
11. $(2x^2+3xy+7y^2)(2x-3y)$
12. $(m^2-3mn-n^2)(m^2+3mn+n^2)$
13. $(x^2-2xy+y^2)(x-y)$
14. $(2x^2-x+5)(x-1)$
15. $(a^2-2)(2a^2-3)$
16. $(x^2+x+1)(x-1)$
17. $(x^2-3x+2)(x+3)$
18. $(x^2-5x+6)(x-1)$
19. $(a^3-a^2x+ax^2-x^3)(a-x)$
20. $(2x^2-x+3)(x^2+2x+1)$
21. $(3ay+7xy-9a)(x^2y+2x-3a)$
22. $(4d^3-2d^2+3d-1)(3d^2-2d-3)$
23. $(p^2q-q^2r^3-r^3s^2+s^2p)(p^2r^2-qr^2)$
24. $(a^2+2ab+b^2)(a^2-2ab+b^2)$
25. $(x^3-5x^2+3x+2)(2x-7)$
26. $(25x^2-10xm+16m^2)(5x+4m)$
27. $(16a^3-12a^2b+4ab^2-15b^3)(a-b)$
28. $(3+4x-7x^2+3x^3)(7-11x)$
29. $(5p^2q^2-7r^2s^2)(5p^2q^2+7r^2s^2)$
30. $(x^3-2x^2+4x-8)(x+2)$
31. $(x^2-12x+16)(\frac{1}{2}x-\frac{1}{4})$
32. $(10m^2n^2+20mnp+25p^2)(.4mn-.6p)$

To find the product
of two factors
first multiply
the numbers
and
[illegible]
the last
numbers × the
first

33. $(6x^3 - 4x^2 + 7x - 3)(7x^2 - 11x - 9)$

34. $(\frac{3}{4}x^3y - \frac{1}{2}x^2y^2 + \frac{2}{3}xy^3)(4x + 6y)$

53. The expression $1 + a + a^2 + a^3 + a^4 + a^5$ is said to be arranged according to the ascending powers of a. The exponent of a in any term is greater than its exponent in any preceding term. The expression $a^5 + a^4 + a^3 + a^2 + a + 1$ is said to be arranged according to the descending powers of a. The exponent of a in any term is less than its exponent in any preceding term.

EXERCISES

Arrange the following expressions according to the ascending or descending powers of some letter:

1. $2y^3 + 3y^2 - 6y - y^4 - 8$

2. $a^2 - 3a^3 + 6a - 3 + a^4$

3. $x - 2x^3 + x^4 - x^2 + 7$

4. $8 - 2c^3 + 5c^2 - 8c + c^4$

5. $3a^2x^3 + 2ax^4 + 2x^5 + 5a^3x^2 + 6a^5$

In the following problems, arrange the terms in multiplier and multiplicand according to the descending powers of some one letter and multiply.

1. $2x^2 - 3x + 5x^3 + x^4 - 6$ by $2 + x$

2. $7 + 6a^2 - 2a + 3a^3$ by $a - 3$

3. $m^3 - 3m^2 + 2m^4 - 7 + 2m$ by $2m - 1$

4. $3b - 4 - b^3 - b^2 - b^4$ by $b^2 - b$

5. $x^2 - 3 + x^3 - 2x$ by $x^2 - x + 2$

6. $k - 2k^3 + k^5 - 3$ by $k + 4$

7. $m^2n^2 + 2m^3n + mn^3 + m^4 + n^4$ by $m^2 - n^2 - mn$

8. $7 + x^2 - 3x + 4x^3$ by $3x + 4x^2 - 2$

9. $a^2 - a + a^3 - 4$ by $2 + a^2 - a$

10. $a^2b^2 + a^4 + b^4$ by $a^4 + b^4 - a^2b^2$

REVIEW EXERCISES

54. 1. $(x+4)(x-10)=?$

2. $(x^2-3x+9)(x+3)=?$

3. If $x=4$ and $y=10$, what does $x+y$ equal? Why? What does $x-y$ equal? Why?

4. How is the product affected by multiplying each factor of $3 \cdot 7$ by 5? What is the product multiplied by if the 3 is multiplied by 4 and the 7 by 3?

5. What is the product multiplied by when both multiplier and multiplicand are multiplied by the same number?

6. How is the product affected if the multiplier is multiplied by k and the multiplicand is multiplied by h?

7. Solve: $12(x-1)-7(x+4)=15$.

8. A merchant bought goods for \$500 and sold them at a gain of 5%. What was the selling price? If he sold them at a loss of 5%, what was the selling price?

9. If in problem 8 the merchant sold the goods at a gain of x per cent, what was the selling price? What was the selling price if he sold them at a loss of x per cent?

10. What was the gain in problem 8? the loss?

11. What was the gain in problem 9? the loss?

12. A merchant sold goods at a gain of 30%, thereby gaining \$120. Let x represent the cost, make an equation, and solve.

13. A merchant sold goods that cost him \$1200 for \$1500. Let x represent the gain per cent, make an equation, and solve.

14. Solve: $6(x-2)=4(x+2)-17$.

15. Solve: $8x - 17 = 6x - (x - 4)$.

16. There are two numbers whose sum is 89, and the larger is 7 more than the smaller. What are the numbers?

17. Separate 2500 into three parts, such that the second part shall exceed the first by 300 and the third part exceed the second by 100.

18. Multiply $8 - 3z^3 + 12z - 10z^2$ by $2z - 6 + z^2$. Check for $z = 1$.

19. From the product of $p^2 - 2p + 4$ and $p + 2$, subtract the product of $p^2 + 2p + 4$ and $p - 2$.

20. If A and B travel in the same direction, A at the rate of x miles an hour and B at the rate of y miles an hour, how many miles will A gain on B in 5 hours?

21. If $x + y = 12$ and $x - y = 6$, what does $2x$ equal? Why? What does $2y$ equal? Why?

22. $\frac{3}{4}x(12x - 8) - \frac{2}{3}x(9x + 12) = ?$

23. A train running x miles an hour makes the trip between two towns in 8 hours. How many miles apart are the towns?

24. Multiply $y^3 - 4y^2 - 8 + 3y$ by $2y - y^2 + 8$.

SUMMARY

SECTION

49. Commutative and associative laws of multiplication.
50. Multiplication of monomials. Law of exponents in multiplication. Examples.
51. Multiplication of a polynomial by a monomial. Examples.
52. Multiplication of a polynomial by a polynomial. Examples.
53. Arrangement according to ascending or descending powers.
54. Review exercises.

CHAPTER X

DIVISION

55. Law of Exponents in Division. Division is the inverse of multiplication. In multiplication

Multiplier × multiplicand = product.

In division (exact)

Divisor × quotient = dividend.

In multiplication the two factors are known and the product is to be found. In division the product (dividend) and one of the factors (divisor) are known and the other factor (quotient) is to be found.

Since in multiplication the exponent of any number (letter or figure) in the product equals the sum of the exponents of that number in multiplier and multiplicand, it follows that the exponent of any number in the dividend must equal the sum of the exponents of that number in quotient and divisor. Hence, the exponent of any number in the quotient equals its exponent in the dividend less its exponent in the divisor. For example, in multiplication $2^3 \cdot 2^2 = 2^5$. Hence $2^5 \div 2^2 = 2^3$. $a^4 \cdot a^2 = a^6$. Hence $a^6 \div a^2 = a^4 = a^{6-2}$.

We have, then, the following rule:

The exponent of a number in the quotient is found by subtracting its exponent in the divisor from its exponent in the dividend.

This may be expressed briefly in a formula as follows:

$$\mathbf{a^m \div a^n = a^{m-n}}$$

56. Division of a Monomial by a Monomial.

ILLUSTRATIVE EXERCISE

Divide $32a^3x^2$ by $8ax$

$$\frac{32a^3x^2}{8ax} = 4a^2x$$

Since a product is divided when any one of its factors is divided, the division is performed by dividing 32 by 8, $\frac{32}{8} = 4$; a^3 by a, $\frac{a^3}{a} = a^2$; and x^2 by x, $\frac{x^2}{x} = x$. The quotient is $4a^2x$.

Section 29 gives the law of signs for division. What is this law?

EXERCISES

Divide mentally:

1. $3 \cdot 8$ by 2
2. $4 \cdot 8 \cdot 9$ by 3
3. 3^5 by 3^2
4. $2^3 \cdot 5^2$ by $2 \cdot 5$
5. $4a^2$ by $-a$ = $-4a$
6. $-6a^2b$ by $2ab$ = $-3a$
7. $8x^2y$ by $-2y$
8. $-22a^2bc$ by $-11ac$
9. $-48b^2x^3y^2$ by $12bx^2y$
10. $34x^5$ by $-17x^3$
11. $84a^2x^6$ by $-7ax^5$
12. 5^{12} by 5^8
13. $-12a^3c^2x^5$ by $-3ac^2x^3$
14. $33a^7x^4$ by $11a^3x^2$
15. $48(a+b)^3$ by $8(a+b)^2$ = $6(a+b)$
16. $42(x-y)^5$ by $-7(x-y)^2$
17. $-15(c-d)^2$ by $(c-d)$
18. $-32x^2(m+n)^5$ by $-4x(m+n)^2$
19. $-75c^4x^2z$ by $25c^3z$
20. $150m^5n^4$ by $-50m^2n^3$
21. $105m^2(x+1)^3$ by $-15m$

22. $-256r^3(x+y)^3$ by $64(x+y)^2$

23. $\frac{3}{4}p^3r^2$ by $-3pr$

24. $3 \cdot 5x^4y^2$ by $5x^2y$

25. $\frac{2}{3}r^5s^3$ by $3r^4s^2$

57. Division of a Polynomial by a Monomial.

According to section 51,

$$(a+b+c)x = ax+bx+cx$$

Therefore,
$$\frac{ax+bx+cx}{x} = a+b+c$$

That is,
$$\frac{ax+bx+cx}{x} = \frac{ax}{x}+\frac{bx}{x}+\frac{cx}{x}$$

Since each is equal to $a+b+c$.

To divide a polynomial by a monomial:

Divide each term of the polynomial by the monomial and connect the quotients thus obtained by the proper signs.

EXERCISES

Divide mentally:

1. $4 \cdot 6 + 5 \cdot 10 - 2 \cdot 8$ by 2

2. $4a+6b+2c$ by 2

3. $3a^2-4a^3$ by a

4. $24a^3x-36a^2x^3$ by $12ax$

5. $6b^3y^2+12b^3y^3-18b^7y^4$ by $6by$

6. $4a^2bx-12a^3b^2x^3+16ab^3x^2$ by $4ax$

7. $12m^4-14m^2$ by $2m^2$

8. $15x^2y+20x^2y^3$ by $5x^2y$

9. $27x^2y^2z+36x^2y^3z^2$ by $9xy^2$

10. $48a^2b^5c^7 - 24a^2b^3c^2$ by $8ab^3c^2$

11. $14a^2 - 35a^3$ by $7a^2$

12. $24a^3bc - 30a^2bc^3 + 42ab^3c^2$ by $6abc$

13. $20x^3y^5 + 24x^3y^4 - 32a^2x^2y^2$ by $4x^2y$

14. $18a^2m^3x^5 - 27b^2m^2x^3 - 36c^2m^4x^3$ by $9m^2x^3$

15. $8x^2y^3 + 24x^3y^2 - 32x^4y^4$ by $-8xy^2$

16. $-12a^3c^2y^5 - 28a^4c^3y^4 - 4a^2c^2y^5$ by $-4a^2c^2y^3$

17. $m^{10} + m^8 + m^6$ by m^2

18. $1.2x^5 + 2.4x^2 + 3x$ by $.6x$

19. $\frac{2}{5}r^2s - \frac{4}{5}rs^2 + \frac{6}{5}rs$ by $2rs$

20. $-3.6m^3n^2 + 2.4m^4n^3 - 6m^2n^2$ by $1.2mn$

21. $81x^5(a+b)^3 - 54x^2(a+b) + 36x^3(a+b)^2$ by $-9x(a+b)$

22. $34a^3(x+y)^3 + 51a^2(x+y)^2 + 85a^5(x+y)^4$ by $17a^2(x+y)^2$

17

23. $3x^2y^3 + 5x^3y^2 + 10xy^4$ by $\frac{2}{3}xy$

24. $.8x^4 + .4x^3 - 1.2x^2$ by $.4x$

25. $\frac{1}{2}m^3 - \frac{1}{3}m^2 - \frac{1}{4}m$ by $\frac{2}{3}m$

58. Division of a Polynomial by a Polynomial.

ILLUSTRATIVE EXERCISE

Multiply $x^2 - x + 6$ by $x + 2$

$$\begin{array}{r}
x^2 - x + 6 \quad\quad\;\; \\
x + 2 \quad\quad\;\; \\
\hline
x^3 - x^2 + 6x \quad\quad\;\; \\
2x^2 - 2x + 12 \\
\hline
x^3 + x^2 + 4x + 12
\end{array}$$

We will now divide $x^3 + x^2 + 4x + 12$ by $x^2 - x + 6$.

$$\begin{array}{l|l} x^3 + x^2 + 4x + 12 & \underline{x^2 - x + 6} \\ \underline{x^3 - x^2 + 6x} & \quad x + 2 \\ \quad + 2x^2 - 2x + 12 & \\ \quad \underline{+ 2x^2 - 2x + 12} & \end{array}$$

It is convenient to place the divisor at the right and then divide the first term of the divisor into the first term of the dividend. This gives x. Multiplying the divisor by x gives $x^3 - x^2 + 6x$. (This is the same as the first partial product in the multiplication.) We subtract this product from the dividend and have $2x^2 - 2x + 12$ as a first remainder. Dividing the first term of this by the first term of the divisor gives $+ 2$. Multiplying the divisor by $+ 2$, we obtain $2x^2 - 2x + 12$. (This is the same as the second partial product in the multiplication.) Subtracting this from the first remainder leaves no remainder.

It is evident from the above work and explanation that the process in division is the converse of that in multiplication.

The division may be checked by substituting numbers for the letters and dividing the numbers thus obtained for dividend and divisor. Or it may be checked by multiplication. In exact division,

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient}$$

From this,

$$\textit{dividend} = \textit{divisor} \times \textit{quotient}$$

In division with a remainder,

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

From this,

$$\textit{dividend} = \textit{quotient} \times \textit{divisor} + \textit{remainder}$$

We have the following steps for dividing by a polynomial:

I. Arrange dividend and divisor according to the ascending or descending powers of some letter.

II. Divide the first term of the dividend by the first term of the divisor to obtain the first term of the quotient.

III. Multiply the divisor by this term of the quotient and subtract the result from the dividend.

IV. Divide the first term of this remainder by the first term of the divisor to obtain the second term of the quotient.

V. Multiply the divisor by this second term of the quotient and subtract the result from the first remainder.

VI. Continue this process until the first term of the divisor is not contained in the first term of the remainder.

ILLUSTRATIVE EXERCISE

Divide $a^3 - 8a^2 + 19a - 14$ by $a - 2$

Solution:

$$
\begin{array}{lrl}
 & a^3 - 8a^2 + 19a - 14 & \big|\ \underline{\ a - 2 \qquad\qquad} \\
\text{First partial product} & \underline{a^3 - 2a^2 \qquad\qquad\quad} & \ \ a^2 - 6a + 7 \\
\text{First partial dividend} & -6a^2 + 19a - 14 & \\
\text{Second partial product} & \underline{-6a^2 + 12a \qquad\quad} & \\
\text{Second partial dividend} & +7a - 14 & \\
\text{Third partial product} & \underline{7a - 14} &
\end{array}
$$

Check by letting $a = 1$. Dividend $= -2$; divisor $= -1$; quotient $= 2$.

Since $\dfrac{\text{dividend}}{\text{divisor}} = \dfrac{-2}{-1} = 2$, the work checks.

EXERCISES

Divide and check:

1. $x^2 + 5x + 6$ by $x + 2$
2. $x^2 - x - 6$ by $x - 3$
3. $x^2 + 3x - 10$ by $x + 5$
4. $x^2 - 1$ by $x - 1$

$$\begin{array}{l|l} x^2 \qquad -1 & \underline{x - 1} \\ \underline{x^2 - x} & x + 1 \\ \quad +x - 1 & \\ \quad \underline{+x - 1} & \end{array}$$

5. $a^3 - 1$ by $a - 1$
6. $n^3 + 1$ by $n + 1$
7. $k^2 - 64$ by $k + 8$
8. $4x^2 - 25$ by $2x + 5$
9. $25n^2 - 4$ by $5n - 2$
10. $x^3 - 8$ by $x - 2$
11. $8b^3 + 27$ by $2b + 3$
12. $64c^2 - 25d^2$ by $8c - 5d$
13. $x^4 - y^4$ by $x^2 - y^2$
14. $g^2 + 11g + 24$ by $g + 3$
15. $6x^2 - 7x - 3$ by $2x - 3$
16. $a^3 - 64$ by $a - 4$
17. $8x^3 - 27$ by $2x - 3$
18. $a^6 + b^6$ by $a^2 + b^2$
19. $6x^2 + 7x - 5$ by $2x - 1$
20. $x^2 - ax + bx - ab$ by $x - a$.
21. $6y^3 - 4y^2 - 3y + 2$ by $3y - 2$
22. $m^2 - 12m + 35$ by $m - 7$
23. $24y^2 + 52yz - 32z^2$ by $3y + 8z$
24. $a^3 - a^2b + ab^2 - b^3$ by $a - b$
25. $a^2 + 3a - 28$ by $a + 7$
26. $x + 5x^2 - 6$ by $5x + 6$
27. $6m^2 + 5m - 6$ by $3m - 2$
28. $a^3 + 3a^2b + 3ab^2 + b^3$ by $a + b$
29. $a^4 + a^2b^2 + b^4$ by $a^2 + ab + b^2$
30. $12a^2 - 38a + 28$ by $6a - 7$
31. $x^3 - 7x^2 - 12x + 84$ by $x - 7$
32. $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$ by $a^2 - 2ab + b^2$
33. $x^5 + y^5$ by $x + y$

34. $3x - 5x^2 + 6x^4 - 2x^3 - 6$ by $3x^2 - x + 2$

35. $4m + 2m^3 - 7m^2 + m^4$ by $m^2 + 3m - 4$

36. $x^3 + 4x^2 + 5x + 8$ by $x + 3$

$$
\begin{array}{r|l}
x^3+4x^2+5x+8 & x+3 \\ \cline{2-2}
x^3+3x^2 \phantom{{}+5x+8} & x^2+x+2 \\ \cline{1-1}
x^2+5x+8 & \\
x^2+3x \phantom{{}+8} & \\ \cline{1-1}
2x+8 & \\
2x+6 & \\ \cline{1-1}
2 &
\end{array}
$$

The last remainder does not contain the first term of the divisor, x, and as a rule the division is not continued after such a remainder has been found. Hence,

$$\frac{x^3 + 4x^2 + 5x + 8}{x + 3} = x^2 + x + 2 + \frac{2}{x + 3}$$

Check by putting $x = 1$

$$\frac{1 + 4 + 5 + 8}{1 + 3} = 1 + 1 + 2 + \frac{2}{4}$$

$$\tfrac{18}{4} = 4 + \tfrac{2}{4}$$

In some of the following problems the division is not exact.

Divide and check:

37. $3x^2 + 8xy + 14y^2$ by $3x + 2y$

38. $2y^3 + 9y^2 + 14y + 60$ by $2y^2 + 7y + 5$

39. $42a^2 - 35a - 10$ by $14a - 7$

40. $14b^2 - 21b - 60$ by $2b - 21$

41. $3p^2 - 3p^3 + p^4 + 11p - 6$ by $2 - 3p + p^2$

42. $36a^2 + 42ax - 2x^2$ by $3a - 4x$

43. $x^3 + y^3$ by $x - y$

44. $6a^4 + 7a^3 + 7a^2 + 16a + 4$ by $3a^2 + 2a + 4$

45. $x^4 + y^4$ by $x + y$

46. $8s^2 - 5s^3 + 2s^4 - 8s + 8$ by $2s^2 - 3s - 1$

47. $6x^4 - x^3 - 14x^2 + x + 8$ by $2x^2 - x - 3$

48. $x^4 - 9x^2 - 6xy - y^2$ by $x^2 + 3x + y$

49. $x^6 + y^6$ by $x^2 - y^2$

50. $a^5 - 2a^4 + 6a^2 - 7a - 5$ by $a^2 + a - 2$

51. $y^5 - 2y^4 + 2y^3 + 3y^2 - 5y + 3$ by $y^2 - 2y + 3$

52. $x^5 - y^5$ by $x + y$

53. $x^4 + x^2y^2 + 2y^4$ by $x^2 - xy + y^2$

54. $x^5 - 6x^4 + 5x^3 - x^2 + 6x + 5$ by $x^2 - 6x + 5$

55. $a^4 + 4a^3 - 11a^2 + 5a - 15$ by $a^2 + 6a - 5$

56. $x^6 - 2x^3 - 4x + 2$ by $x + 2$

57. $2x^4 + 5x^3 - 37x^2 + 44x + 84$ by $x^2 + 5x - 6$

58. $a^5 + a^4b + 2a^3b^2 - b^5$ by $a^2 + ab + b^2$

59. $4a^4 - 4a^3 + a^2 - 16a + 15$ by $2a^2 + 3a + 5$

60. $9y^3 + 2y + 4$ by $3y^2 - 2y + 2$

REVIEW EXERCISES

59. **1.** $4 \cdot (-6) \cdot (-\frac{1}{3}) \cdot \frac{3}{4} \cdot (-10) = ?$

2. $3ax \cdot (-4a^2) \cdot \frac{2}{3}x \cdot (-5x^2) \cdot 20ax = ?$

3. $73m^2n^2 - 12mn \cdot (-3n) + 2m \cdot 6n = ?$

4. $(x + y)(x + y) - (x - y)(x - y) = ?$

5. A man is 32 years old. How old was he four years ago? x years ago?

6. Multiply the sum of $3a^2 - 4ax + 3x^2$ and $5ax - 2a^2 + x^2$ by $-4ax$.

7. Divide $(x^2 + xy + y^2)(x - y) + (x^2 - xy + y^2)(x + y)$ by x.

8. Divide $27x^3 - 108x^2 + 144x - 64$ by $3x - 4$.

9. Solve the problem from the manuscript of Ahmes given in the historical note (section 15); *viz.*, "Heap, its seventh, its whole, it makes 19. What is it?"

10. If $x = 2$ and $y = -3$, find the value of
$$2x^2 + 4[xy - x(3x - 4y) - 3y(4x + 2y)].$$

11. The difference between two numbers is 3 and their product exceeds the square of the smaller by 36. What are the numbers?

12. The sum of a number, its half, and its third is 44. What is the number?

13. The ones' digit of a number is 2 less than twice the tens' digit. If the order of the digits be reversed, the resulting number exceeds the original number by 18. What is the number?

14. $[(3x^2 - 7x + 12) - (2x^2 + x - 3)] \div [(x^2 - 4x + 3) \div (x - 1)] = ?$

15. Divide $4a - 3a^2 + 2a^3 - 3$ by $2a^2 + 3 - a$.

16. Solve: $3(4x - 2) = 2(x + 12)$.

17. Solve: $30x - 3(4x - 2) + 5x = 10x + 45$.

18. A rectangular field is 20 rods longer than wide. If the width were increased by 5 rods, the area would be increased by 200 square rods. What are the dimensions of the field?

19. Mt. Hood, Oregon, is 1355 feet less than twice as high as Mt. Washington, New Hampshire, and Mt. McKinley, Alaska, is 1986 feet less than twice as high as Mt. Hood. The combined height of Mt. Hood and Mt. Washington is 2949 feet less than the height of Mt. McKinley. What is the height of each?

20. Solve: $10(x - 1) + 12(2x - 6) = 20 + 15(x - 3)$.

7. 9-10-11-12-13-14-26

21. What number multiplied by $x - 6$ will give $x^3 - x^2 - 24x - 36$?

22. What number divided by $x^2 - 4x - 12$ will give $x + 3$?

23. A man has $3.50 in dimes and quarters. He has 17 coins in all. How many has he of each?

24. Multiply $12x^4 - 3x^3 + 19x^2 - 5x + 4$ by $3x^3 - 4x^2 + 5x - 4$ and check the result by letting $x = 2$.

25. Divide $2a^5 - 7a^4b - 2a^3b^2 + 8a^2b^3 - 13ab^4$ by $2a - 3b$ and check the result by letting $a = 2$ and $b = 1$.

26. From a field 20 rods longer than wide a square block at one end was sold, leaving 5 acres. What were the dimensions of the field? 60 rd 40 rds,

27. Divide $155 among A, B, C, and D so that A and B together receive $40, C receives twice as much as A, and D receives three times as much as B.

28. Two towns are 250 miles apart. How long will it take a train running x miles an hour to make the trip between them?

29. Two towns are 300 miles apart. One train makes an average of x miles an hour and a faster train an average of y miles an hour. How much longer will it take the first train to make the trip between the towns than it takes the second train?

SUMMARY

SECTION

55. Law of exponents in division.
56. Division of a monomial by a monomial. Exercises.
57. Division of a polynomial by a monomial. Exercises.
58. Division of a polynomial by a polynomial. Exercises.
59. Review exercises.

CHAPTER XI

SIMPLE EQUATIONS—CONTINUED

60. A term is **integral** with respect to any letter when that letter has a positive integral exponent, and does not appear in the denominator.

The term $\frac{a^2bx^3}{c}$ is integral with respect to a, b, and x, but is not integral with respect to c. A term that is integral with respect to all of its letters is called an **integral term.**

A polynomial is **integral** with respect to a letter when all of its terms are integral with respect to that letter.

The polynomial $\frac{x^2}{a} + bx + a^2$ is integral with respect to x and b, but is not integral with respect to a.

An equation is **integral** with respect to a letter when its two members are integral with respect to that letter.

The equation $3x^2 - 5x + 6 = 18$ is integral with respect to x.

61. The **degree** of an integral term is the sum of the exponents of its literal factors.

Thus $a^2b^3x^5$ is of the 10th degree. What is the degree of z^2y^3? $3c^3x^5$? $5ax^2yz^2$?

The **degree** of an integral term with respect to particular letters is the sum of the exponents of those letters.

The term a^2x^3 is of the third degree in x. $3x^2y$ is of the third degree in x and y. c^2x and $5x$ are of the first degree in x.

Give the degree of the following in x:

$3x$, $5x^2$, ax, $7x^3$, abx^2, $6x^5$, bx^4.

The **degree** of a polynomial is that of its term of highest degree.

Thus, $x^3 + 5x^2 + 3x$ is of the third degree.

$xy + x$ is of the first degree in x and the second degree in x and y.

The **degree** of an equation is that of its term of highest degree.

$x^2 + 2x = 17$ is an equation of the second degree.

$x^2y = 12$ is an equation of the third degree in x and y.

62. A **simple equation** is an equation of the **first** degree.

Simple equations in one or two unknowns are often called **linear equations.**

$3x + 5x + 5 = 4x + 17$ is a simple equation.

EXERCISES

Solve, mentally when possible:

1. $5s = 30$ = 6
2. $x + 3 = 8$ = 5
3. $y - 5 = 3$ +8
4. $y + 8 = 5$ −3
5. $\frac{s}{4} = 6$ = 24 (8.)
6. $11 - 2y = 6$ 5/2
7. $3x + 4 = 5$
8. $6k - 2 = 7$
9. $4z + 3 = z + 12$
10. $x - \frac{x}{3} = 6$ = 9
11. $2x - 5 = x + 6$
12. $3x + 4 = 5x - 6$
13. $a - 3 = 6 - 2a$
14. $x + 8 = 10 - 5x$
15. $3y - 7 = y + 7$
16. $x + \frac{x}{3} + \frac{x}{4} = \frac{19}{6}$

    Hint: Multiply each member by 12.
17. $3x + (2x - 8) = 17$
18. $5x - (3x + 4) = 14$

19. $6x - 18 = 2(x + 5)$

20. $7y - (6 - y) = 5y + 9$

21. $12x + 3(3x - 5) = 2(2x + 7)$

22. $36x - 4(x - 7) = 12x + 3(7 - 2x)$

23. $ax = 6$

24. $(a + 2)x = 5$

25. $ax + 2x = 10$

26. $ax + bx = c$ 1.10

Will you won't you

Why not

R

27. One of two numbers is 5 larger than the other. Their sum is 51. What are the numbers?

28. The sum of two numbers is 80 and their difference is 6. Find the numbers.

29. The sum of two numbers is 37 and the larger exceeds 5 times the smaller by 1. What are the numbers?

30. The sum of three consecutive integers is 39. What are the numbers?

31. The sum of three consecutive even numbers is 30. Find the numbers.

32. The sum of a number, its half and its third is 66. What is the number?

33. Divide 128 into three parts such that the second part shall be 2 more than the first and the third part shall be 3 less than the first.

34. A student's expenses for a year were $270. He paid $5 more for books than for laundry and his board cost him $5 less than 7 times as much as his laundry. What was the amount of each item of expense?

35. In a certain school the number of pupils under 14 years of age is 5 greater than the number over 18 years of age. The number from 14 to 18 inclusive is

4 times the number in the other two classes. If there are 125 pupils in all, how many are there in each class?

36. The United States Navy, in 1913, contained 3 times as many third class cruisers as first class, and one less of the first class than of the second class. The total number of cruisers was 26. How many were there of each class?

37. A merchant sold a number of pairs of shoes at \$4 a pair and 5 more than that number of pairs at \$3 a pair. He received \$85 for all. How many pairs did he sell at each price?

38. A lady bought 10 pounds of meat for \$1.80. For some she paid 15 cents a pound and for the remainder 20 cents a pound. How much of each kind did she buy?

39. A father is 30 years older than his son. Five years ago the father was 3 times as old as the son. What are their ages?

Suggestion: If x represents the son's age, what represents the father's age? What was the father's age 5 years ago? What was the son's age 5 years ago? three times the son's age 5 years ago? What is said of the father's age 5 years ago and 3 times the son's age at that time?

40. A man is 10 years more than twice as old as his son. Four years ago he was 3 times as old as his son. What is the age of each?

41. Find a number whose ones' digit is 2 greater than its tens' digit and which is equal to 7 times its ones' digit.

42. The ones' digit of a number is 3 more than the tens' digit. The number is 4 times the sum of its digits. What is the number?

41-49-53-55-56-57-
58

43. The ones' digit of a number is twice the tens' digit. If 36 be added to the number, the order of the digits is reversed. What is the number?

44. The standing armies of Germany, Great Britain and France number approximately 1,434,000. The standing army of France exceeds twice that of Great Britain by 52,000 and the German army exceeds that of France by 60,000. Find the number in each army.

45. The product of two consecutive odd numbers is 22 less than the square of the greater. What are the numbers?

46. The product of two consecutive even numbers is 40 greater than the square of the smaller. What are the numbers?

47. If a certain number is multiplied by a number 5 greater than itself, the resulting product is 2 greater than the square of a number 2 greater than the given number. What is the number?

48. A gallon of alcohol weighs .7 pounds less than a gallon of turpentine and 2 pounds less than a gallon of milk. Three gallons of alcohol, 5 gallons of turpentine, and 2 gallons of milk weigh 73.5 pounds. What is the weight of a gallon of each?

49. A cubic foot of mercury weighs approximately 139 pounds more than a cubic foot of lead and 356 pounds less than a cubic foot of gold. Five cubic feet of gold weigh 208 pounds less than 4 cubic feet of lead and 4 cubic feet of mercury. Find the weight of a cubic foot of each.

50. Detroit, according to the last U. S. Census, had 36,000 more than 5 times as many inhabitants as Des Moines, and Milwaukee had 30,000 more than 4 times as many as Des Moines. The three cities had a population of 926,000. What was the population of each?

51. The area of Oregon is 16,000 square miles less than that of Wisconsin and Illinois combined. The area of Idaho is 12,800 square miles less than that of Oregon, and the area of Illinois is 600 square miles more than that of Wisconsin. The four states contain 293,300 square miles. What is the area of each?

52. A man bought calves at $16, cows at $70, and sheep at $10 each. He bought half as many cows as calves and three times as many sheep as cows, paying $660 for all. How many of each did he buy?

53. A man has $3000 bearing interest, some at 5% and the remainder at 6%. The annual interest on both sums is $155. How much is invested at each rate?

54. During the year ending June 30, 1912, the imports of coffee into the United States amounted to 28 million dollars more than twice those of fruit and nuts. The imports of raw silk were 25 million dollars more than those of fruits and nuts. The value of these three classes of imports was approximately 223 million dollars. What was the value of the imports in each class?

55. A launch makes a trip of 44 miles in 5 hours running 3 hours on a lake and 2 hours down a river. If the rate on the river is 2 miles an hour greater than on the lake, what is the rate on the lake?

Suggestion: If x represents the rate per hour on the lake, what represents the rate per hour on the river? What represents the distance traveled on the lake? on the river? What two expressions are there for the total distance?

56. An automobile party made a trip of 96 miles in 6 hours running at the rate of 20 miles an hour in the country and 8 miles an hour in the towns. What was the time in the country? What was the distance traveled in the country?

57. A man who can row 5 miles an hour in still water rows up a stream for 3 hours and then rows back to his starting point in 2 hours? At what rate does the stream flow?

58. A man rows up a stream flowing 2 miles an hour for 4 hours and then back to his starting point in 2 hours. How fast can the man row in still water?

59. The Municipal Building, New York, is twice as high as Trinity Church tower, and the Metropolitan Life Insurance building is 140 feet less than 3 times as high as Trinity Church tower. Four times the height of the Metropolitan building is 5 times the height of the Municipal building. What is the height of each building?

60. The Erie Canal is 27 miles more than 4 times as long as the Suez, and the Miami, joining Cincinnati and Toledo, is 4 miles more than 3 times as long as the Suez. Twice the length of the Erie diminished by twice the length of the Miami is 44 miles less than 3 times the length of the Suez. Find the length of each canal.

SUMMARY

SECTION

60. Definition of integral terms, etc.
61. Definition of degree of term, etc.
62. Simple equations. Definition. Problems.

CHAPTER XII

SPECIAL PRODUCTS AND QUOTIENTS. FACTORS

63. The **factors** of a number are the numbers which multiplied together give that number. (See section 5.) Thus, 3, 7, and 5 are factors of 105. Since

$$(a+3)(a-2)=a^2+a-6$$

$a+3$ and $a-2$ are factors of a^2+a-6.

A **prime** factor is a factor which cannot be separated into other factors.

64. An algebraic expression is **rational** with **respect** to any letter, when, in its simplest form, it contains no indicated root of that letter. For example,

$$x^2-\frac{3a}{x}+7x-23$$

is rational with respect to x.

By the factors of a number, unless otherwise stated, prime rational integral factors are meant.

65. Monomial Factors.

$$x(a+b)=xa+xb$$

Therefore, by the definition of factors, x and $a+b$ are the factors of $xa+xb$. That is, $xa+xb=x(a+b)$.

ILLUSTRATIVE EXERCISES

1. What are the factors of $cy+dy$?

Solution:

It is evident that y and $c+d$ multiplied together give

$cy + dy$. Hence, y and $c + d$ are the factors of $cy + dy$. That is,

$$cy + dy = y(c + d)$$

The eye should be trained to detect monomial factors at a glance. In order that a monomial may be a factor of an algebraic expression, it must be a factor of each term of the expression. Stating it conversely, if a number appears in every term of an algebraic expression, it is a factor of the expression.

2. Factor $a^2b + a^2c + a^2d + a^2x$

Solution:

Here a^2 occurs in every term and is a factor. We have

$$a^2b + a^2c + a^2d + a^2x = a^2(b + c + d + x)$$

Factor the following by inspection:

1. $2a + 2b$
2. $ca + cb$
3. $xy - yz$
4. $2x + 2$
5. $3a - 12$
6. $5m - 10n + 15$
7. $7x - 14y - 21z$
8. $5ab + 15ac - 20ay$
9. $3axy - 2dxy + 4mxy$
10. $3c^2 - 6cy$
11. $4a^3b - 6a^2b^2 + 8a^3b$
12. $3a^3 - 6a^2$
13. $5a^4 - 10a^2b$
14. $8m^3n + 4mn^3$
15. $14x^2y^2 - 7xy$
16. $24c^3a^2 + 8c^2a^5$
17. $42sr^5 - 14s^4r^2$
18. $ax^5 + a^2x^4 + a^3x^3$
19. $3a^5y - 6a^3y^3 - 9ay^4$
20. $4a^3b^3c^3 - 6a^2b^4c^3 + 8a^3b^3c^4$
21. $y^{10} - y^8 + y^6 + 2y^4$
22. $36x^3y^3 - 24x^2y^3 + 12x^3y^2$
23. $8x - 12y + 24z$
24. $3a^2x^2y^3 - 6a^3xy^3 + 9ax^3y^2$
25. $14a^3x + 28a^3y + 35a^3z$

26. $a(x+y)+b(x+y)=(a+b)(x+y)$

27. $x(u+v)+y(u+v)$

28. $3a(c+d)-2x(c+d)$

29. $b(3m+2)-4c(3m+2)$

30. $a^2(x+1)-3a(x+1)+4(x+1)$

SUGGESTION. In factoring any algebraic expression, monomial factors should be removed first. The remaining factors may frequently be factored again. Examples of this will be given in what follows.

66. Polynomials Factored by Grouping the Terms.

ILLUSTRATIVE EXERCISES

1. Multiply $a+b$ by $c+d$.

Solution:

Performing the multiplication by multiplying $a+b$ by c and then by d and adding the partial products,

$$\begin{aligned}(c+d)(a+b) &= c(a+b)+d(a+b)\\ &= (ac+bc)+(ad+bd)\\ &= ac+bc+ad+bd\end{aligned}$$

If the problem be reversed, that is, if the factors of $ac+bc+ad+bd$ are sought, we reverse the process, taking c out of the first two terms and d out of the last two terms. Thus,

$$\begin{aligned}ac+bc+ad+bd &= (ac+bc)+(ad+bd)\\ &= c(a+b)+d(a+b)\\ &= (c+d)(a+b)\end{aligned}$$

2. Multiply $a+b$ by $x+y+z$.

Solution:

Multiplying $a+b$ by x, then by y, and then by z, and adding the partial products,

$$\begin{aligned}(x+y+z)(a+b) &= x(a+b)+y(a+b)+z(a+b)\\ &= (ax+bx)+(ay+by)+(az+bz)\\ &= ax+bx+ay+by+az+bz\end{aligned}$$

If we are required to factor $ax + bx + ay + by + az + bz$, we reverse the process. Thus,

$$\begin{aligned} ax+bx+ay+by+az+bz &= (ax+bx)+(ay+by)+(az+bz) \\ &= x(a+b)+y(a+b)+z(a+b) \\ &= (x+y+z)(a+b) \end{aligned}$$

Whenever the terms of a polynomial can be so grouped that all the groups contain a common factor, the polynomial can be factored in this way. Notice that in the first example above, each of the two groups contains the factor $a + b$, and in the second, each of the three groups contains the factor $a + b$.

Caution. In grouping the terms of a polynomial, strict attention must be given to the principle that if the terms are enclosed in a parenthesis preceded by a minus sign, the sign of each term so enclosed must be changed.

3. Factor $x^2 + ax - xy - ay$.

Solution:

$$\begin{aligned} x^2 + ax - xy - ay &= (x^2 + ax) - (xy + ay) \\ &= x(x + a) - y(x + a) \\ &= (x - y)(x + a) \end{aligned}$$

EXERCISES

Factor the following:

1. $3x(a + c) + y(a + c)$
2. $2b(x - y) - c(x - y)$
3. $3s(a + s) + k(a + s)$
4. $ab + bc + ad + cd$
5. $bk + ak + cb + ac$
6. $hm - 3m + 2h - 6$
7. $2x^3 - 3x^2 + 4x - 6$
8. $6ax + 10 + 3a^2x + 5a$
9. $m^3 - m^2 - 2m + 2$
10. $12x + 9y - 8xy - 6y^2$
11. $x^2 - 4x + xy - 4y$
12. $a^3 - a^2 + a - 1$
13. $m^2 - m - a + am$
14. $1 + x + y + xy$
15. $3a^3 - 15a - 2a^2b + 10b$

16. $3x^3 - 5x^2 + 6x - 10$
17. $2y^3 - 3y^2 - 6y + 9$
18. $a^2b + b + a^2c + c$
19. $mn - n - m + 1$
20. $ab - 3a - 2b + 6$
21. $x^3y^2 - y^2z^3 - x^3d + z^3d$
22. $ax^2 + 5x^2 + 3ax + 15x + 2a + 10$
23. $a^3 - 5a^2 - 2a^2b + 10ab + ab - 5b$
24. $xz + x - 5yz - 5y - 6z - 6$
25. $abx + bcx + cax + aby + bcy + cay$
    Hint. Group the first three terms and the last three terms.
26. $21b - 5a + 3ab - 2ac - 14c - 35$
    Hint. Rearrange before grouping.
27. $14ay + 21y^2 - 4a^2x - 6axy + 8a + 12y$
28. $x^3 - 7x^2 + 9x + 2ax^2 - 14ax + 18a$
29. $x^5 + x^4 + x^3 + x^2 - 2x - 2$
30. $ax + by + ay + bz + az + bx + cx + cy + cz$

67. The products of certain types of binomials and of a binomial and a trinomial occur so often that it is important for the student to recognize these types and to be able to write the products by inspection.

68. Products of the form $(x + a)\,(x + b)$

ILLUSTRATIVE EXERCISES

1. Multiply $x + 3$ by $x + 5$.

Solution:

$$\begin{array}{l} x + 3 \\ \underline{x + 5} \\ x^2 + 3x \\ \underline{\phantom{x^2 + {}} 5x + 15} \\ x^2 + 8x + 15 = x^2 + (3 + 5)\,x + 3 \cdot 5 \end{array}$$

2. Multiply $x + 7$ by $x - 4$.

Solution:

$$\begin{array}{l} x + 7 \\ \underline{x - 4} \\ x^2 + 7x \\ \underline{\quad\; - 4x - 28} \\ x^2 + 3x - 28 = x^2 + [7 + (-4)]x + 7 \cdot (-4) \end{array}$$

3. Multiply $x - 3$ by $x - 7$.

Solution:

$$\begin{array}{l} x - 3 \\ \underline{x - 7} \\ x^2 - 3x \\ \underline{\quad\; - 7x + 21} \\ x^2 - 10x + 21 = x^2 + [-3 + (-7)]x + (-3) \cdot (-7) \end{array}$$

In each of the three examples above, the first term of the product (x^2) is the square of the term (x) common to the two binomials. The second term is the product of the common term by the sum of the second terms of the binomials. The third term is the product of the second terms of the binomials.

4. Multiply $x + 2$ by $x + 5$ by inspection.

Solution:

$$\begin{aligned}(x + 2)(x + 5) &= x^2 + (2 + 5)x + 2 \cdot 5 \\ &= x^2 + 7x + 10\end{aligned}$$

5. Multiply $m + 8$ by $m - 2$ by inspection.

Solution:

$$\begin{aligned}(m + 8)(m - 2) &= m^2 + (8 - 2)m + 8 \cdot (-2) \\ &= m^2 + 6m - 16\end{aligned}$$

EXERCISES

Write the following products by inspection:

1. $(a+2)(a+3)$

2. $(y+3)(y-7)$

3. $(x-7)(x+2)$

4. $(a-2)(a+4)$

5. $(m+1)(m+4)$

6. $(y+10)(y-3)$

7. $(a+7)(a+9)$

8. $(z-8)(z-3)$

9. $(x-3)(x+5)$

10. $(s+3)(s-11)$

11. $(m-3)(m+2)$

12. $(m-4)(m-12)$

13. $(m-4)(m+12)$

14. $(v^2+7)(v^2+2)$

15. $(c^2-2)(c^2-6)$

16. $(d-16)(d-3)$

17. $(b+7)(b-12)$

18. $(rs-2)(rs-7)$

$$\begin{aligned}(rs-2)(rs-7) &= (rs)^2+[-2+(-7)]\,rs+(-2)(-7)\\ &= r^2s^2-9rs+14\end{aligned}$$

19. $(ax-4)(ax+12)$

20. $(xy-18)(xy+10)$

21. $(mn-7)(mn+8)$

22. $(pq-5)(pq-15)$

23. $(ab-14)(ab-2)$

24. $(x+a)(x+b)$

25. $(y+c)(y+d)$

26. $(x-a)(x-b)$

27. $(y-c)(y+d)$

28. $(x-a)(x+b)$

29. $(xy-s)(xy+r)$

30. $(3x-2)(3x+4)$

$$\begin{aligned}(3x-2)(3x+4) &= (3x)^2+(-2+4)\,3x+(-2)\cdot 4\\ &= 9x^2+6x-8\end{aligned}$$

31. $(2x+3)(2x+5)$

32. $(4x-3)(4x+1)$

33. $(5x+2)(5x-3)$

34. $(4x-3)(4x-1)$

35. Multiply 35 by 32.

Solution:

$$\begin{aligned}32\cdot 35 &= (30+2)(30+5)\\ &= 30^2+(2+5)\,30+2\cdot 5\\ &= 900+210+10\\ &= 1120\end{aligned}$$

Find in this way the following products:

36. $24 \cdot 23$
37. $34 \cdot 37$
38. $27 \cdot 26$
39. $43 \cdot 46$
40. $72 \cdot 78$
41. $54 \cdot 59$
42. $42 \cdot 47$
43. $83 \cdot 84$
44. $65 \cdot 63$
45. $71 \cdot 74$
46. $105 \cdot 106$
47. $112 \cdot 108$

69. Factors of Expressions of the Form $\mathbf{x^2 + ax + b}$

What are the factors of $a^2 + 5a + 6$? (See Ex. 1, section 68.)

What are the factors of $m^2 + 5m + 4$? (See Ex. 5, section 68.)

(What are the factors of $x^2 + 2x - 15$? (See Ex. 9, section 68.)

What are the factors of $y^2 + 7y - 30$? (See Ex. 6, section 68.)

What are the factors of $m^2 + 8m - 48$? (See Ex. 13, section 68.)

What relation does the last term in each of the above examples bear to the last terms of the factors?

What relation does the numerical coefficient of the middle term bear to the last terms of the factors?

The product of $x + 3$ and $x + 5$ is $x^2 + (3 + 5)x + 3 \cdot 5$. Or, using a more general form,

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

The factors of $x^2 + (3 + 5)x + 3 \cdot 5$ are $x + 3$ and $x + 5$. The factors of $x^2 + (a + b)x + ab$ are $x + a$ and $x + b$. That is,

$$\mathbf{x^2 + (a + b)x + ab = (x + a)(x + b)}$$

ILLUSTRATIVE EXERCISES

1. Factor $x^2 + 7x + 10$.

Solution:

In order that this trinomial may be the product of two binomials of the form $x + a$ and $x + b$, the sum of a and b must be 7 and the product of a and b must be 10. These conditions are met if a is 2 and b is 5. Therefore,

$$x^2 + 7x + 10 = (x + 2)(x + 5)$$

2. Factor $a^2 - 8a + 12$.

Solution:

Here, two numbers must be found whose sum is -8 and whose product is 12. The required numbers are -2 and -6. Therefore,

$$a^2 - 8a + 12 = (a - 2)(a - 6)$$

3. Factor $m^2 - 7m - 18$.

Solution:

Here, two numbers are required whose sum is -7 and whose product is -18. They are evidently -9 and 2 and we have

$$m^2 - 7m - 18 = (m - 9)(m + 2)$$

What is true of the signs of the factors, if in the trinomial to be factored all the terms are positive? if the middle term is negative and the last term is positive? if the last term is negative?

EXERCISES

Factor the following expressions, by inspection when possible:

1. $x^2 + 3x + 2$

2. $x^2 - 4x + 3$

3. $x^2 + 13x + 40$

4. $x^2 - 14x + 40$

5. $x^2 - 3x - 40$
6. $x^2 + 3x - 40$
7. $y^2 - 6y + 8$
8. $a^2 - 3a - 28$
9. $m^2 + 5m + 4$
10. $x^2 - 2x - 15$
11. $s^2 - 5s - 14$
12. $c^2 + 10c + 25$
13. $c^2 + 8c - 48$
14. $b^2 - 15b + 54$
15. $x^2 - 5x - 84$
16. $a^2 - 5a - 66$
17. $t^2 + 3t - 88$
18. $y^2 - y - 12$
19. $r^2 - 2r - 80$
20. $k^2 + 17k + 72$
21. $x^2 + 12x - 108$
22. $a^2b^2 + 6ab - 16$
23. $c^2d^2 - 4cd - 21$
24. $x^2y^2 - 5xy - 14$
25. $(a + b)^2 + 5(a + b) + 6$
26. $(x + y)^2 - 12(x + y) + 20$
27. $(x + 3)^2 - 4(x + 3) - 60$
28. $(a - 2)^2 + 6(a - 2) - 27$
29. $x^4 - 6x^2 - 16$
30. $y^6 + 10y^3 + 21$
31. $z^4 + 2z^2 - 120$
32. $(2k)^2 + 5(2k) + 6$
33. $24 - 5n - n^2 = (8 + n)(3 - n)$
34. $24 + 11y + y^2$
35. $15 + 2c - c^2$
36. $12 + h - h^2$
37. $33 - 8z - z^2$
38. $ax^2 - 12ax + 32a = a(x^2 - 12x + 32)$
$= a(x - 8)(x - 4)$
39. $bm^2 + 2bm - 35b$
40. $cr^2 - 4cr - 45c$
41. $2m^2 + 12m + 16$
42. $5ax^2 + 40ax + 75a$

43. $4x^3 - 28x^2 - 72x$

44. $12a + 18ay + 6ay^2$

45. $30bc + 9bcd - 3bcd^2$

70. Quotients of the form $\dfrac{x^2 + (a + b)x + ab}{x + a}$

Began

We have seen in section 68 that

$$(x + 3)(x + 5) = x^2 + 8x + 15$$

Therefore, since division is the inverse of multiplication,

$$\frac{x^2 + 8x + 15}{x + 3} = x + 5, \text{ or } \quad \frac{x^2 + 8x + 15}{x + 5} = x + 3$$

Also, since $(x + 7)(x - 4) = x^2 + 3x - 28$, we have

$$\frac{x^2 + 3x - 28}{x + 7} = x - 4, \text{ or } \quad \frac{x^2 + 3x - 28}{x - 4} = x + 7$$

Also, since $(x - 3)(x - 7) = x^2 - 10x + 21$, we have

$$\frac{x^2 - 10x + 21}{x - 3} = x - 7, \text{ or } \quad \frac{x^2 - 10x + 21}{x - 7} = x - 3$$

EXERCISES

Write by inspection the following quotients:

1. $\dfrac{a^2 + 8a + 15}{a + 3}$

2. $\dfrac{x^2 - 5x - 14}{x - 7}$

3. $\dfrac{a^2 + 2a - 8}{a + 4}$

4. $\dfrac{y^2 - 4y - 21}{y + 3}$

5. $\dfrac{m^2 + 7m - 30}{m + 10}$

6. $\dfrac{z^2 - 11z + 24}{z - 8}$

7. $\dfrac{x^2 + 2x - 15}{x + 5}$ n − 3

8. $\dfrac{a^2 + 16a + 63}{a + 7}$ a + 9

9. $\dfrac{n^2 - 3n - 18}{n - 6}$ n + 3

10. $\dfrac{p^2 + 5p - 66}{p + 11}$

11. $\dfrac{s^2 - 3s - 88}{s - 11}$

12. $\dfrac{c^2 + 2c - 63}{c + 9}$

13. $\dfrac{k^2 - 17k + 60}{k - 5}$

14. $\dfrac{m^2 + 8m - 48}{m + 12}$

15. $\dfrac{b^2 - 15b - 34}{b - 17}$

16. $\dfrac{a^2x^2 + 8ax - 48}{ax + 12}$

17. $\dfrac{x^2y^2 - 8xy - 180}{xy - 18}$

18. $\dfrac{a^2b^2 - 12ab - 28}{ab + 2}$

19. $\dfrac{x^2 + (c + d)x + cd}{x + d}$

20. $\dfrac{y^2 - (a + b)y + ab}{y - a}$

21. $\dfrac{3 + 4x + x^2}{3 + x}$

22. $\dfrac{8 - 2x - x^2}{2 - x}$

23. $\dfrac{12 + 4x - x^2}{6 - x}$

24. $\dfrac{24 + 11x + x^2}{8 + x}$

25. $\dfrac{36 - 15b + b^2}{12 - b}$

26. $\dfrac{72 - x - x^2}{9 + x}$

71. Under the type form $(x + a)(x + b) = x^2 + (a + b)x + ab$, there are two special cases of sufficient importance to be considered by themselves.

72. The Square of a Binomial.

$$\begin{aligned}(x+4)^2 &= (x+4)(x+4)\\ &= x^2+(4+4)x+4\cdot 4\\ &= x^2+8x+16\end{aligned}$$

$$\begin{aligned}(x-4)^2 &= (x-4)(x-4)\\ &= x^2-(4+4)x+4\cdot 4\\ &= x^2-8x+16\end{aligned}$$

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a^2+(b+b)a+b\cdot b\\ &= a^2+2ab+b^2\end{aligned}$$

The square of a binomial is equal to the square of the first term plus or minus twice the product of the absolute values of the two terms plus the square of the second term.

This principle may be stated in the following form:

$$\mathbf{(a+b)^2 = a^2+2ab+b^2}$$

$$\mathbf{(a-b)^2 = a^2-2ab+b^2}$$

When do we take plus twice the product? when minus twice the product?

ILLUSTRATIVE EXERCISES

1. Write by inspection the square of $2x+3y$.

Solution:

$$(2x+3y)^2 = (2x)^2+2\cdot 2x\cdot 3y+(3y)^2 = 4x^2+12xy+9y^2$$

2. Write by inspection the square of $7x-5$.

Solution:

$$(7x-5)^2 = (7x)^2-2\cdot 7x\cdot 5+5^2 = 49x^2-70x+25$$

EXERCISES

Write by inspection the following squares:

1. $(x+2)^2$
2. $(y-4)^2$
3. $(m+7)^2$
4. $(k+11)^2$
5. $(c+3)^2$
6. $(x-8)^2$
7. $(2x+3)^2$
8. $(2r-10)^2$
9. $(4m+7)^2$
10. $(5c-3)^2$
11. $(2y-5)^2$
12. $(xy+3)^2$
13. $(cx+2)^2$
14. $(5c-8x)^2$
15. $(am+4)^2$
16. $(6z-5)^2$
17. $(ax-b)^2$
18. $(cs+r)^2$
19. $(4bc-3x)^2$
20. $(12+7x)^2$
21. $(8-11xz)^2$
22. $(x+\frac{2}{3})^2 = x^2+2\cdot\frac{2}{3}\cdot x+(\frac{2}{3})^2 = x^2+\frac{4}{3}x+\frac{4}{9}$
23. $(m+\frac{1}{2})^2$
24. $(x-\frac{2}{5})^2$
25. $(c-\frac{3}{7})^2$

The squares of arithmetical numbers may be found in the same manner.

$$92^2=(90+2)^2=90^2+2\cdot 90\cdot 2+2^2=8100+360+4=8464$$

$$89^2=(90-1)^2=90^2-2\cdot 90\cdot 1+1^2=8100-180+1=7921$$

EXERCISES

Find in this manner the following squares, performing the work mentally:

1. 31^2
2. 52^2
3. 64^2
4. 84^2
5. 39^2
6. 79^2
7. 58^2
8. 99^2
9. 45^2
10. 102^2
11. 97^2
12. 75^2
13. 98^2
14. 105^2
15. 81^2
16. 999^2

73. Factors of Trinomial Squares. We have seen that the square of a binomial is a trinomial. The factors, then, of such a trinomial are the binomial used twice. The factors of x^2+4x+4 are $x+2$ and $x+2$. (See exercise 1, section 72.) That is,

$$x^2+4x+4=(x+2)(x+2)$$

18½ 1

It is necessary to know when a trinomial is such a square. Consider the following squares:

$$a^2 + 2ab + b^2 = (a + b)^2 \quad (1)$$

$$a^2 - 2ab + b^2 = (a - b)^2 \quad (2)$$

The left hand members of (1) and (2) are trinomials and are squares. Notice that the first and third terms are **squares** and **positive.** The middle term is plus or minus twice the product of the absolute values of the square roots of the first and third terms.

A trinomial is the square of a binomial if two of its terms are squares and positive and the remaining term is twice the product of the absolute values of their square roots. The binomial is the sum or the difference of two numbers according as the term corresponding to the second term of (1) is plus or minus.

160
2
320

ILLUSTRATIVE EXERCISES

Is $25a^2 - 40a + 16$ a perfect square?

The terms $25a^2$ and 16 are squares and positive and the remaining term $(-40a)$ is minus twice the product of the absolute values of their square roots. That is,

$$-2 \cdot \sqrt{25a^2} \cdot \sqrt{16} = -2 \cdot 5a \cdot 4 = -40a$$

Hence, $25a^2 - 40a + 16 = (5a - 4)(5a - 4) = (5a - 4)^2$

On the other hand, is $9x^2 + 42x + 64$ a perfect square?

The terms $9x^2$ and 64 are squares and positive. But the remaining term $(42x)$ is not twice the product of the absolute values of their square roots, since

$$2 \cdot \sqrt{9x^2} \cdot \sqrt{64} = 2 \cdot 3x \cdot 8 = 48x$$

Hence, $9x^2 + 42x + 64$ is not a perfect square.

EXERCISES

Determine which of the following are perfect squares and factor them:

1. $x^2 - 10x + 25$
2. $m^2 + 22m + 121$
3. $a^2 + 6a + 9$
4. $x^2 - 12x + 36$
5. $9c^2 - 6c + 1$
6. $16x^2 + 8x + 4$
7. $r^2 + 8r + 16$
8. $16x^2 + 16x + 16$
9. $1 - 10y + 25y^2$
10. $49 - 28z + 4z^2$
11. $(a + b)^2 + 2(a + b) + 1$
12. $4x^2 - 4x + 1$
13. $100x^2 + 40x + 4$
14. $4a^2 + 12ab + 9b^2$
15. $49z^2 - 14zy + 4y^2$
16. $6x^2 - 12x + 36$
17. $x^2 + x + \frac{1}{4}$
18. $4x^2 - \frac{4}{3}x + \frac{1}{9}$
19. $m^2 - 1.2m + .36$
20. $.04x^2 + 1.2x + 9.$

74. Completing the Square. Some binomials may be changed into trinomial squares by the addition of a number. This process is called **completing the square.**

ILLUSTRATIVE EXERCISES

1. Add a number to $x^2 - 16x$ such that the resulting expression shall be a trinomial square.

Solution:

The number to be added must be a square such that twice the product of its square root and $\sqrt{x^2}$ shall be $-16x$. Hence, dividing $-16x$ by twice the square root of x^2 gives the square root of the number to be added.

$$\frac{-16x}{2\sqrt{x^2}} = \frac{-16x}{2x} = -8$$

Hence the number to be added to $x^2 - 16x$ to make the resulting trinomial a perfect square is $(-8)^2$, or 64. Adding this, we have

$$x^2 - 16x + 64$$

which is the square of $x - 8$.

2. Make $4m^2 + 24m$ a perfect square.

Solution:

The number to be added must be a square such that twice the product of its square root by $\sqrt{4m^2}$ shall be $24m$. Hence, dividing $24m$ by twice the square root of $4m^2$ will give the square root of the number to be added.

$$\frac{24m}{2\sqrt{4m^2}} = \frac{24m}{4m} = 6$$

Adding 6^2, or 36, we have $4m^2 + 24m + 36$, which is the square of $2m + 6$.

3. Make $9x^2 + 4$ a perfect square.

Solution:

The two terms given are perfect squares and positive. Hence, we must add plus or minus twice the product of their square roots, that is $2 \cdot 3x \cdot 2$, or $12x$. We have then

$$9x^2 + 12x + 4 \text{ or } 9x^2 - 12x + 4$$

EXERCISES

Make each of the following a perfect square:

1. $x^2 - 10x$
2. $x^2 + 4x$
3. $m^2 - 6m$
4. $y^2 + 12y$
5. $k^2 - 18k$ 81
6. $z^2 - 24z$
7. $4x^2 + 25$
8. $4x^2 + 36$
9. $9n^2 + 24n$
10. $25x^2 - 60x$
11. $x^2 + 36$
12. $16y^2 - 40y$
13. $x^2 + 3x$
14. $x^2 + \frac{1}{4}$
15. $36b^2 + 12b$

75. The product of the sum and difference of two numbers.

$$(x + 3)(x - 3) = x^2 + (3 - 3)x + 3 \cdot (-3) = x^2 - 9$$

The sum of the numbers in the coefficient of the second term is zero. In general,

$$(\mathbf{a} + \mathbf{b})(\mathbf{a} - \mathbf{b}) = \mathbf{a}^2 - \mathbf{b}^2$$

The product of the sum and difference of two numbers is equal to the square of the first number minus the square of the second number. Or, more briefly,

The product of the sum and difference of two numbers equals the difference of their squares.

EXERCISES

Find the following products by inspection:

1. $(a+2)(a-2)$
2. $(m-4)(m+4)$
3. $(y+7)(y-7)$
4. $(x-9)(x+9)$
5. $(r+5)(r-5)$
6. $(x-10)(x+10)$
7. $(z+20)(z-20)$
8. $(a+b)(a-b)$
9. $(c-2d)(c+2d)$
10. $(am+8)(am-8)$
11. $(bc-12)(bc+12)$
12. $(3-m)(3+m)$
13. $(4x+5)(4x-5)$
14. $(8y+10)(8y-10)$
15. $(9t-13)(9t+13)$
16. $(ms-pq)(ms+pq)$
17. $(3-11ax)(3+11ax)$
18. $(7s-20)(7s+20)$
19. $(7hk-8x)(7hk+8x)$
20. $(x-.5)(x+.5)$
21. $(abc-2)(abc+2)$
22. $(5-2x)(5+2x)$
23. $(12z+7)(12z-7)$
24. $(5mn-11c)(5mn+11c)$
25. $(2xy+7cd)(2xy-7cd)$
26. $(12xz-11b)(12xz+11b)$
27. $(15-9uv)(15+9uv)$
28. $(21+15x)(21-15x)$
29. $(14rs-13z)(14rs+13z)$
30. $(.4c+.7d)(.4c-.7d)$

31. $(x+y+5)(x+y-5) = [(x+y)+5][(x+y)-5]$
$$= (x+y)^2 - 5^2$$
$$= x^2 + 2xy + y^2 - 25$$

32. $(x+y+7)(x+y-7)$ [illegible]

33. $(a+c-3)(a+c+3)$ [illegible]

34. $(m-n+4)(m-n-4)$ [illegible]

35. $(3a-2-x)(3a-2+x)$ [illegible]

36. $(3x-\frac{2}{3})(3x+\frac{2}{3})$ $9x^2 - \frac{4}{9}$

37. $(x^2+x+1)(x^2+x-1)$ [illegible] $+ x^2 - 1$

38. $(a+b+c)(a+b-c)$

39. $(.9mn-.7k)(.9mn+.7k)$

The above principle may be used in the multiplication of certain arithmetical numbers.

40. Find the product of 77 and 83.

Solution:

$$77 \cdot 83 = (80-3)(80+3) = 80^2 - 3^2 = 6400 - 9 = 6391$$

In this manner find the following products, mentally when possible: [illegible]

41. $93 \cdot 87$
42. $65 \cdot 75$
43. $29 \cdot 31$
44. $38 \cdot 42$
45. $47 \cdot 53$
46. $109 \cdot 91$
47. $79 \cdot 81$
48. $49 \cdot 51$
49. $136 \cdot 124$
50. $88 \cdot 92$
51. $83 \cdot 97$
52. $84 \cdot 76$
53. $37 \cdot 43$
54. $78 \cdot 62$
55. $67 \cdot 73$
56. $123 \cdot 117$

76. Factors of the Difference of Two Squares.

Since $(a+b)(a-b) = a^2 - b^2$, the factors of $a^2 - b^2$ are $a+b$ and $a-b$. That is,

$$\mathbf{a^2 - b^2 = (a+b)(a-b)}$$

If the expression to be factored is the difference of the squares of two numbers, the factors are evidently the sum of the numbers and the difference of the numbers.

ILLUSTRATIVE EXERCISES

1. Factor $x^2 - 16$.

Solution:

Here we have the square of x minus the square of 4. The factors are the sum of x and 4 and the difference of x and 4. Hence,

$$x^2 - 16 = (x + 4)(x - 4)$$

An expression containing more than two terms may sometimes be put in the form of the difference of two squares.

2. Factor $25 - c^2 - 8cd - 16d^2$.

Solution:

$$\begin{aligned} 25 - c^2 - 8cd - 16d^2 &= 25 - (c^2 + 8cd + 16d^2) \\ &= 25 - (c + 4d)^2 \\ &= [5 + (c + 4d)][5 - (c + 4d)] \\ &= (5 + c + 4d)(5 - c - 4d) \end{aligned}$$

EXERCISES

Factor the following expressions, by inspection when possible:

1. $x^2 - 4$
2. $a^2 - 9$
3. $s^2 - 16$
4. $k^2 - 49$
5. $z^2 - 100$
6. $x^2 - 121$
7. $c^2 - 64$
8. $x^2 - m^2$
9. $y^2 - 4a^2$
10. $d^2 - 25a^2$
11. $a^2b^2 - 81$
12. $a^2c^2 - x^2$
13. $4x^2 - 9$
14. $49a^2 - 9x^2$
15. $(a + b)^2 - 9$
16. $(a + b)^2 - y^2$
17. $(x - y)^2 - a^2$
18. $25x^2y^2 - 64c^2$
19. $36x^2 - 121$
20. $49x^2y^2 - 81c^2d^2$

21. $100m^2n^2 - 144b^2$

22. $m^2 - (x - y)^2$

23. $4c^2 - (a - b)^2$

24. $(x - 3)^2 - (y + 2)^2$

25. $(a + b)^2 - (c - d)^2$

26. $a^2 + 2ab + b^2 - c^2$
$= (a^2 + 2ab + b^2) - c^2$
$= (a + b)^2 - c^2 = ?$

27. $x^2 + 2xy + y^2 - 16$

28. $m^2 + 2mn + n^2 - 9y^2$

29. $4a^2 - 12a + 9 - 16z^2$

30. $9y^2 - x^2 + 6x - 9$

31. $x^2 - 2x + 1 - m^2 - 2mn - n^2$
$= (x^2 - 2x + 1) - (m^2 + 2mn + n^2)$
$= (x - 1)^2 - (m + n)^2 = ?$

32. $m^2 + 6m + 9 - 4x^2 + 4x - 1$

33. $4x^2 - 24x + 36 - 16m^2 + 40mn - 25n^2$

34. $5ax^2 - 20a = 5a(x^2 - 4) = 5a(x + 2)(x - 2)$

35. $am^2 - 16a$

36. $9x^3 - 25x$

37. $32mz^2 - 2m$

38. $3 - 27c^2d^2$

39. $x^4 - a^4$

40. $x^8 - y^8$

41. Factor 899.

Solution:

$$899 = 900 - 1 = (30 + 1)(30 - 1) = 31 \cdot 29$$

In this manner factor the following mentally:

42. 399

43. 391

44. 3599

45. 1591

46. 1596

47. 4891

48. 2491

49. $851 = 900 - 49$

50. 884

51. 2451

52. $221 = 225 - 4$

53. 8099

77. Quotients of the form $\frac{a^2 - b^2}{a + b}$ and $\frac{a^2 - b^2}{a - b}$

Since (section 75) $(a + b)(a - b) = a^2 - b^2$, we have

$$\frac{a^2 - b^2}{a + b} = a - b \qquad \text{or} \qquad \frac{a^2 - b^2}{a - b} = a + b$$

EXERCISES

Write by inspection the following quotients:

1. $\frac{a^2-4}{a-2}$
2. $\frac{m^2-9}{m+3}$
3. $\frac{y^2-49}{y+7}$
4. $\frac{x^2-81}{x-9}$
5. $\frac{b^2-25}{b-5}$
6. $\frac{x^2-100}{x-10}$
7. $\frac{z^2-400}{z+20}$
8. $\frac{c^2-4d^2}{c+2d}$
9. $\frac{a^2m^2-64}{am+8}$
10. $\frac{b^2c^2-144}{bc-12}$
11. $\frac{64-m^2}{8+m}$
12. $\frac{16x^2-25}{4x+5}$
13. $\frac{64y^2-100}{8y-10}$
14. $\frac{25-16s^2}{5-4s}$
15. $\frac{a^2b^2c^2-4}{abc+2}$
16. $\frac{(a+c)^2-9}{a+c+3}$
17. $\frac{(a+b)^2-(c+d)^2}{a+b-c-d}$

78. Trinomials of the form $ax^2 + bx + c$.*

ILLUSTRATIVE EXERCISES

1. Multiply $3x+5$ by $2x+3$.

Solution:
$$\begin{aligned}(2x+3)(3x+5) &= 2x(3x+5)+3(3x+5)\\ &= 6x^2+2x\cdot 5+3\cdot 3x+15\\ &= 6x^2+10x+9x+15\\ &= 6x^2+19x+15\end{aligned}$$

---

*There are several methods of factoring expressions of the form $ax^2 + bx + c$. Many teachers prefer the cross-multiplication, or trial method. This is easily presented and may be used instead of the method given. The following method brings the form $ax^2 + bx + c$ under section 69. Factor $6x^2 + 13x + 6$.

Solution: Multiplying by 6 so as to make the first term a square and then dividing by 6 so that the value of the expression remains unchanged, we have

$$6x^2+13x+6=\frac{(6x)^2+13(6x)+36}{6}=\frac{(6x+9)(6x+4)}{6}$$

$$=\frac{6x+9}{3}\cdot\frac{6x+4}{2}=(2x+3)\ (3x+2)$$

2. Multiply $cx + d$ by $ax + b$.

Solution:

$$\begin{aligned}(ax+b)(cx+d) &= ax(cx+d) + b(cx+d)\\ &= acx^2 + adx + bcx + bd \qquad (1)\\ &= acx^2 + (ad+bc)x + bd\end{aligned}$$

If our problem is to factor $6x^2 + 19x + 15$, we may reverse our steps in the first problem solved above, first separating $19x$ into $10x + 9x$, and then factoring by grouping.

Solution:
$$\begin{aligned}6x^2 + 19x + 15 &= 6x^2 + 10x + 9x + 15\\ &= 2x(3x+5) + 3(3x+5)\\ &= (2x+3)(3x+5)\end{aligned}$$

In the above problems the first and last terms of the product contain as factors all the terms of the two binomials. But the two middle terms in (1) above also contain as factors all the terms of the two binomials. Hence, the product of the first and last terms of the product is equal to the product of the two middle terms in (1).

In factoring expressions of this kind, then, our first effort will be to separate the middle term into two parts whose product equals the product of the first and last terms. In the problem above, $19x$ was separated into $10x$ and $9x$ and $10x \cdot 9x = 6x^2 \cdot 15$.

3. Factor $5x^2 + 13x + 6$.

Solution:

The product of $5x^2$ and 6 is $30x^2$. We must find two numbers whose sum is $13x$ and whose product is $30x^2$. The required numbers are $10x$ and $3x$. The given expression may then be factored as follows:

$$\begin{aligned}5x^2 + 13x + 6 &= 5x^2 + 10x + 3x + 6\\ &= 5x(x+2) + 3(x+2)\\ &= (5x+3)(x+2)\end{aligned}$$

4. Factor $3x^2 - 5x - 22$.

Solution:

The product of $3x^2$ and -22 is $-66x^2$. We must find, if possible, two numbers whose sum is $-5x$ and whose product is $-66x^2$. As the product is negative, the two numbers will have unlike signs. $-11x$ and $6x$ fulfill both conditions. Hence,

$$\begin{aligned} 3x^2 - 5x - 22 &= 3x^2 - 11x + 6x - 22 \\ &= x(3x - 11) + 2(3x - 11) \\ &= (x + 2)(3x - 11) \end{aligned}$$

Check: Let $x = 1$. Then $3 - 5 - 22 = 3 \cdot (-8) = -24$

EXERCISES

Factor the following and check results:

1. $2x^2 + 3x + 1$
2. $2x^2 + 5x + 2$
3. $2x^2 + 7x + 3$
4. $2x^2 + 9x + 4$
5. $2x^2 - 3x + 1$
6. $2x^2 - 5x + 2$
7. $2a^2 - 7a + 3$
8. $2m^2 + 5m + 3$
9. $2n^2 + 7n + 6$
10. $2r^2 + 9r + 9$
11. $2c^2 - c - 6$
12. $2d^2 - 3d - 9$
13. $3x^2 + 5x + 2$
14. $3y^2 + 8y + 4$
15. $3b^2 - b - 2$
16. $3x^2 + x - 2$
17. $3x^2 + 10x + 3$
18. $3s^2 + 8s - 3$
19. $3d^2 + 11d + 6$
20. $3z^2 + 14z + 8$
21. $3x^2 + 10x - 8$
22. $2x^2 + 11x + 15$
23. $2x^2 + 13x + 20$
24. $2x^2 + 13x + 21$
25. $2b^2 - b - 21$
26. $2r^2 - 19r + 35$
27. $4x^2 - 2x - 6$
28. $4x^2 - 4x - 15$
29. $4x^2 + 24x + 35$
30. $6x^2 + 23x + 20$
31. $5m^2 + 22m + 8$
32. $5x^2 - 8x + 3$

15c

33. $6x^2 + 11x + 4$

34. $7x^2 - 9x + 2$

35. $10x^2 + 11x + 3$

36. $8c^2 - 26c + 15$

37. $5x^2 - 23x + 12$

38. $6c^2 + 7c - 10$

39. $7c^2 + 20c - 3$

40. $5x^2 - 19x + 12$

41. $12m^2 + 11m + 2$

42. $10y^2 - 23y + 12$

43. $11y^2 - y - 10$

44. $6z^2 - 5z - 21$

45. $17x^2 + 32x - 4$

46. $14m^2 + 5m - 1$

47. $9z^2 - 36z + 32$

48. $12x^2 - 8x - 15$

49. $10y^2 + 21y - 27$

50. $12k^2 - 32k + 16$
$= 4(3k^2 - 8k + 4)$
$= 4(3k - 2)(k - 2)$

51. $10a^2 + 19a - 15$

52. $3 - 2x - 21x^2$

53. $16 + 34x + 15x^2$

54. $12am^2 - 16am - 3a$

55. $6a^3 + 5a^2 - 56a$

56. $50x^3 - 50x^2 - 72x$

57. $20m^2 - 63mn + 36n^2$

58. $30x^2y^2 + 29xy - 24$

79. The Sum of Two Cubes.

ILLUSTRATIVE EXERCISE

1. Multiply $a^2 - ab + b^2$ by $a + b$.

Solution:

$$\begin{array}{llll}
a^2 - ab + b^2 & & & \\
\underline{a + b} & & & \\
a^3 - a^2b + ab^2 & & & \\
\underline{\quad + a^2b - ab^2 + b^3} & & & \\
a^3 \qquad\qquad\quad + b^3 & & &
\end{array}$$

We see that if a binomial $(a + b)$ consisting of the sum of two numbers, and a trinomial $(a^2 - ab + b^2)$ consisting of the sum of the squares of the two numbers minus their product be multiplied together, the product is the sum of the cubes of the two numbers.

2. Multiply $25x^2 - 20x + 16$ by $5x + 4$.

Solution:

Here, the two numbers are $5x$ and 4. In the trinomial we have $25x^2 = (5x)^2$; $16 = 4^2$; and $20x = 4 \cdot 5x$. Therefore,

$$(25x^2 - 20x + 16)(5x + 4) = (5x)^3 + 4^3 = 125x^3 + 64$$

EXERCISES

Write the following products by inspection:

1. $(a^2 - a + 1)(a + 1)$ a^3+1
2. $(x^2 - 2xy + 4y^2)(x + 2y)$ x^3+8y^3
3. $(m^2 - 2m + 4)(m + 2)$ m^3+8
4. $(4k^2 - 2kx + x^2)(2k + x)$ $8k^3+x^3$
5. $(4x^2 - 6x + 9)(2x + 3)$ $8x^3+27$
6. $(3y + 1)(9y^2 - 3y + 1)$ $27y^3+1$
7. $(6r + 5)(36r^2 - 30r + 25)$ $216r^3+125$
8. $(4q^2 - 4q + 4)(2q + 2)$ $8q^3+8$
9. $(4m^2 - 10m + 25)(2m + 5)$ $8m^3+125$
10. $(9 - 3x + x^2)(3 + x)$ $27+x^3$
11. $(4 + 3x)(16 - 12x + 9x^2)$ $64+27x^3$
12. $(1 - 4h + 16h^2)(1 + 4h)$ $1+64h^3$
13. $(36 - 6s + s^2)(6 + s)$ $216+s^3$
14. $(4c^2 - 12c + 36)(2c + 6)$ $8c^3+216$
15. $(5 + m)(25 - 5m + m^2)$ $125+m^3$
16. $(7a + 2)(49a^2 - 14a + 4)$ $343a^3+8$
17. $(9a^2x^2 - 6ax + 4)(3ax + 2)$ $27a^3x^3+8$
18. $(64x^2 - 56xy + 49y^2)(8x + 7y)$ $512x^3+343y^3$
19. $(16x^2y^2 - 20xyz + 25z^2)(4xy + 5z)$ $64x^3y^3+125z^3$
20. $(9mn + 10xy)(81m^2n^2 - 90mnxy + 100x^2y^2)$ $727m^3n^3+$ $1000x^3y$

80. Factors of the Sum of Two Cubes.

Since $(a+b)(a^2-ab+b^2)=a^3+b^3$, it follows that the factors of a^3+b^3 are $a+b$ and a^2-ab+b^2. That is,

$$\mathbf{a^3+b^3=(a+b)(a^2-ab+b^2)}$$

This expressed in words is

A binomial that is the sum of the cubes of two numbers can be separated into two factors, one of which is the sum of the two numbers and the other the sum of the squares of the two numbers minus their product.

ILLUSTRATIVE EXERCISE

Factor $27a^3+125b^3$.

Solution:

This is the sum of the cubes of $3a$ and $5b$. Hence

$$\begin{aligned}27a^3+125b^3&=(3a+5b)[(3a)^2-3a\cdot 5b+(5b)^2]\\&=(3a+5b)(9a^2-15ab+25b^2)\end{aligned}$$

EXERCISES

Factor the following, by inspection when possible:

1. x^3+1
2. x^3+8
3. a^3+b^3
4. a^3+27b^3
5. $27a^3+1$
6. $8m^3+27b^3$
7. $1+8b^3$
8. $125z^3+64$
9. $125x^3+64y^3$
10. $216r^3+8$
11. m^3+64k^3
12. $k^3p^3+27x^3y^3$
13. $8m^3+125n^3$
14. $27x^3+64$
15. $3x^3+81s^3$
    $=3(x^3+27s^3)$
16. $216+a^3$
17. $4m^3n^3+500$
18. $8c^3+216$
19. $15a^3b^3+120m^3n^3$
20. a^3+b^6
21. x^6+y^6

32/672

81. Quotients of the form $\frac{a^3 + b^3}{a + b}$ and $\frac{a^3 + b^3}{a^2 - ab + b^2}$

Since (section 79), $(a^2 - ab + b^2)(a + b) = a^3 + b^3$,

$$\therefore \frac{a^3 + b^3}{a^2 - ab + b^2} = a + b \quad \text{or} \quad \frac{a^3 + b^3}{a + b} = a^2 - ab + b^2.$$

EXERCISES

Write the following quotients by inspection:

1. $\frac{x^3 + y^3}{x + y}$
2. $\frac{x^3 + 27}{x^2 - 3x + 9}$
3. $\frac{a^3x^3 + b^3}{ax + b}$
4. $\frac{x^3 + 8}{x + 2}$
5. $\frac{8x^3 + 27y^3}{2x + 3y}$
6. $\frac{x^6 + y^6}{x^2 + y^2}$
7. $\frac{a^3 + 216}{a + 6}$
8. $\frac{8a^3 + 125}{2a + 5}$
9. $\frac{27c^3 + 64y^3}{9c^2 - 12cy + 16y^2}$

82. The Difference of Two Cubes.

ILLUSTRATIVE EXERCISES

1. Multiply $a^2 + ab + b^2$ by $a - b$.

Solution:

$$\begin{array}{l} a^2 + ab + b^2 \\ \underline{a - b \qquad\qquad\quad} \\ a^3 + a^2b + ab^2 \\ \underline{\qquad - a^2b - ab^2 - b^3} \\ a^3 \qquad\qquad\qquad\quad - b^3 \end{array}$$

We have here multiplied a trinomial $(a^2 + ab + b^2)$, consisting of the sum of the squares of two numbers plus their product, by a binomial $(a - b)$ consisting of the difference of two numbers. This comes under the preceding type if ~~we~~ consider $a - b$ as the sum of a and $-b$. That is, $x - y = x + (-y)$. We then have

$$[a + (-b)][a^2 - a\cdot(-b) + (-b)^2] = a^3 + (-b)^3$$
$$= a^3 - b^3,$$

or,

$$(a - b)(a^2 + ab + b^2) = a^3 - b^3$$

2. Multiply $16a^2 + 12ab + 9b^2$ by $4a - 3b$.

Here $4a - 3b$ is the difference between the numbers $4a$ and $3b$. In the trinomial $16a^2 = (4a)^2$; $9b^2 = (3b)^2$; and $12ab = 4a\cdot 3b$. Therefore,

$$(16a^2 + 12ab + 9b^2)(4a - 3b) = (4a)^3 - (3b)^3$$
$$= 64a^3 - 27b^3$$

EXERCISES

Write the following products by inspection:

1. $(x^2 + x + 1)(x - 1)$
2. $(y^2 + y + 1)(y - 1)$
3. $(a^2 + ab + b^2)(a - b)$
4. $(4m^2 + 6m + 9)(2m - 3)$
5. $(16y^2 + 4y + 1)(4y - 1)$
6. $(4k^2 + 10k + 25)(2k - 5)$
7. $(9s^2 + 21s + 49)(3s - 7)$
8. $(c - 4d)(c^2 + 4cd + 16d^2)$
9. $(25 + 5x + x^2)(5 - x)$
10. $(8x - 3)(64x^2 + 24x + 9)$
11. $(81r^2 + 18rs + 4s^2)(9r - 2s)$
12. $(4a^2 + 14ab + 49b^2)(2a - 7b)$
13. $(25x^2 + 10x + 4)(5x - 2)$
14. $(100x^2 + 10xy + y^2)(10x - y)$

15. $(x^2y^2 + 4xyz + 16z^2)(xy - 4z)$
16. $(144 + 36y^2 + 9y^4)(12 - 3y^2)$
17. $(4a^2 - 1)(16a^4 + 4a^2 + 1)$
18. $(81a^2c^2 + 99acx + 121x^2)(9ac - 11x)$
19. $(.04x^2 + .2x + 1)(.2x - 1)$
20. $(.5 - .3m)(.25 + .15m + .09m^2)$

83. Factors of the Difference of Two Cubes.

Since (section 82), $(a - b)(a^2 + ab + b^2) = a^3 - b^3$, the factors of $a^3 - b^3$ are $a - b$ and $a^2 + ab + b^2$. That is,

$$\mathbf{a^3 - b^3 = (a - b)(a^2 + ab + b^2)}$$

This expressed in words is,

A binomial which is the difference of the cubes of two numbers can be separated into two factors, one of which is the difference of the two numbers, and the other the sum of the squares of the two numbers plus their product.

ILLUSTRATIVE EXERCISE

Factor $27x^3 - 125y^3$.

Solution:

This is the difference of the cubes of $3x$ and $5y$. Hence,

$$27x^3 - 125y^3 = (3x - 5y)[(3x)^2 + 3x \cdot 5y + (5y)^2]$$
$$= (3x - 5y)(9x^2 + 15xy + 25y^2)$$

EXERCISES

Factor the following, by inspection when possible:

1. $a^3 - 1$
2. $r^3 - 8$
3. $p^3 - 27x^3$
4. $1 - 8a^3$
5. $8y^3 - 125$
6. $8 - 27y^3$
7. $64u^3 - 125y^3$
8. $27r^3 - 64x^3y^3$
9. $125 - 216z^3$
10. $m^3n^3 - 343$

11. $27 - 8a^3p^3$
12. $125c^3 - 8m^3n^3$
13. $27 - 8a^3b^3c^3$
14. $a^3b^3c^3 - d^3s^3$
15. $x^3y^3z^3 - 8c^3$
16. $(a + x)^3 - 8$
17. $(x + 2y)^3 - x^3$
18. $1 - (a - b)^3$
19. $27 - (m - n)^3$
20. $1000x^3 - 729y^3$
21. $5x^3 - 40$
$= 5(x^3 - 8)$
$= 5(x - 2)(x^2 + 2x + 4)$
22. $ax^3 - ay^3$
23. $4 - 32m^3n^3$
24. $5r^2s^3 - 135r^2$
25. $40b - 5bu^3x^3$
26. $2ab^3 - 54a$
27. $7xy^3 - 189x$

84. Quotients of the form $\dfrac{a^3 - b^3}{a - b}$ and $\dfrac{a^3 - b^3}{a^2 + ab + b^2}$

Since (section 82), $(a - b)(a^2 + ab + b^2) = a^3 - b^3$, we have

$$\frac{a^3 - b^3}{a^2 + ab + b^2} = a - b \quad \text{or} \quad \frac{a^3 - b^3}{a - b} = a^2 + ab + b^2$$

EXERCISES

Write the following quotients by inspection:

1. $\dfrac{x^3 - y^3}{x - y}$ $x - y^2$
2. $\dfrac{x^3 - 8}{x - 2}$

$^3 - 27$

7. $\dfrac{8x^3 - 343}{2x - 7}$
8. $\dfrac{x^6 - y^6}{x^2 - y^2}$

$125a^3 - 8$

85. No general rule or method can be given for factoring algebraic expressions. The following suggestions will be helpful:

I. Remove all monomial factors first.

II. If an expression is the difference of two even powers, treat it first as the difference of two squares. For example,

$$\begin{aligned} x^6 - y^6 &= (x^3 - y^3)(x^3 + y^3) \\ &= (x - y)(x^2 + xy + y^2)(x + y)(x^2 - xy + y^2) \end{aligned}$$

III. Polynomials consisting of an even number of terms can probably be grouped by twos so as to show a common factor, or can be expressed as the difference of two squares.

Familiarity with the several types is the only key to readiness in factoring and this can come only with study and practice.

86. Miscellaneous exercises in factoring.

Factor the following:

1. $mn^2 - mn + m$
2. $x^2 + 13x + 42$
3. $x^2 + 2x - 48$
4. $m^2 - 9m + 20$
5. $q^3 - 3c$

13. $a^2x^2 - 12ax + 32$
14. $r^2s + 3r^2s^2 - 5rs^2$
15. $p^4q^2 + 4p^2$
16. $x^2 - 196$
17. x^2n^2

25. $20r^3 - 60r^2 + 45r$
26. $16a^2x^2 - 25m^2$
27. $5m^2 - 10mn$
28. $28y^4 + 7y^2z^2$
29. $c^2 + 17c - 84$
30. $2 - x - x^2$
31. $2 + 3y + y^2$
32. $18c^3 - 9c$
33. $x^2 + 17xy + 70y^2$
34. $a^2 + 10ab - 39b^2$
35. $8 + x^3$
36. $27 - m^3n^3$
37. $25a^2b - 10ab + b$
38. $r^3 + 64$
39. $8 - x^3$
40. $81a^2b^2 - 64m^2n^2$
41. $625r^2 - 225s^2$
42. $24 - 81a^3$
43. $216p^3 - 27f^3$
44. $s^3 - 6s^2t + 2st^2$
45. $m^2n^2 - 4mn - 60$
46. $36x^2 + 60x + 25$
47. $a^2x^2 + 9ax + 14$
48. $x^2 + 8x - 180$
49. $x^2y - 9xy - 22y$
50. $8x^4 + 8x^3y + 2x^2y^2$
51. $a^2b + b + a^2c + c$
52. $2x^2 + 5x + 2$
53. $6m^2 - 11m + 5$
54. $125d^3 + 8k^3$
55. $125c^3d^3 - 216$
56. $ab - 3a - 2b + 6$
57. $3h^2 + 7h + 2$
58. $2a^2 + 9a + 10$
59. $64an^3 - 27ax^3$
60. $8x^4 + 4x^3m + 2x^2$
61. $49m^2n^2 - 70mn + 25$
62. $100g^2h^2 - 81a^2$
63. $x^2y^2 - 3cxy - 10c^2$
64. $1 + 6a - 7a^2$
65. $\frac{1}{25}a^2 - \frac{1}{16}x^2$
66. $24u^5v^3 - 12u^3v + 42u^4v^2$
67. $\frac{4}{81} - \frac{25}{16}m^2n^2$
68. $3a^3 - 24d^3$
69. $x^4 - a^4$
70. $(a+b)^2 + 7(a+b) + 10$
71. $2 + 16m^3n^3$
72. $x^2 + ax + 2x + 2a$
73. $729x^3 - 1$
74. $12x^2 + 5x - 2$
75. $5b^2 - 79b - 16$
76. $x^2 - ax + cx - ac$
77. $8x^2 + 15x - 2$
78. $40m^3n - 135n^4$

79. $216z^3 - 27s^3$

80. $x^2y^2z^2 - 13xyz + 30$

81. $m^6 - n^6$

82. $54r^3s^3 + 2$

83. $8 - (a - b)^3$

84. $a^2b^2c^2 - 19abc + 90$

85. $7m^2 + 84mn + 252n^2$

86. $(x + y)^3 - 64$

87. $a^3x + a^3y + x + y$

88. $1000 - 216r^3s^3$

89. $12x^2 + 29x + 10$

90. $7a^2 + 36a + 5$

91. $a^4 - 6a^2 - 16$

92. $x^4y^4 - 8x^2y^2 + 16$

93. $6a^2c^2 - 23ac + 20$

94. $4x^2 - 20x + 25 - 9y^2$

95. $6m^2n - 46mn - 72n$

96. $12x^2 - 7xy + y^2$

97. $32x^3z^3 + 108m^3$

98. $1 - 49a^2 + 14ax - x^2$

99. $3x^3 + 2x^2 - 3x - 2$

100. $5x^2 - 19xy - 4y^2$

101. $12m^3 - 16am^2 - 3a^2m$

102. $288 - 48x + 2x^2$

103. $27ax^2 - 57axy + 6ay^2$

104. $x^2 + (a + b)x + ab$

105. $2z^2 - (2a + b)z + ab$

106. $ax^2 - 64a$

107. $x^8 - 256y^8$

108. $5a^6 - 320m^3$

109. $(x - y)^2 - 6(x - y) - 40$

110. $27w^3x + 36w^2x^2 + 81wx^3$

111. $56a^2n^4 - 14a^3n^3 - 42a^4n^2$

112. $(x + 3y)^2 - (m - 4n)^2$

113. $a^2 + 2ab + b^2 - x^2 - 2ax - a^2$

114. $2a - 2b + ad - bd + 2c + cd$

115. $x^2 + 4x + 4 - x^2y - 4xy - 4y$

116. $r^2 - 10r + 25 - m^2 + 6m - 9$

SUMMARY

SECTION

63. Factors. Prime factor.
64. Rational expression.
65. Monomial factors.
66. Polynomials factored by grouping the terms.
67. General statement.
68. Products of the form $(x+a)(x+b)$.
69. Factors of expressions of the form x^2+ax+b.
    $x^2+(a+b)x+ab=(x+a)(x+b)$.
70. Quotients of the form $\dfrac{x^2+(a+b)x+ab}{x+a}$.
71. Statement concerning two special cases of trinomials.
72. The square of a binomial.
    $(a+b)^2=a^2+2ab+b^2$.
    $(a-b)^2=a^2-2ab+b^2$.
73. Factors of trinomial squares.
    $a^2+2ab+b^2=(a+b)(a+b)$.
    $a^2-2ab+b^2=(a-b)(a-b)$.
74. Completing the square.
75. The product of the sum and difference of two numbers.
    $(a+b)(a-b)=a^2-b^2$.
76. Factors of the difference of two squares.
    $a^2-b^2=(a+b)(a-b)$.
77. Quotients of the form $\dfrac{a^2-b^2}{a-b}$ and $\dfrac{a^2-b^2}{a+b}$.
78. Factors of trinomials of the form ax^2+bx+c.
79. The sum of two cubes.
80. Factors of the sum of two cubes.
    $a^3+b^3=(a+b)(a^2-ab+b^2)$.
81. Quotients of the form $\dfrac{a^3+b^3}{a+b}$ and $\dfrac{a^3+b^3}{a^2-ab+b^2}$.
82. The difference of two cubes.
83. Factors of the difference of two cubes.
    $a^3-b^3=(a-b)(a^2+ab+b^2)$.
84. Quotients of the form $\dfrac{a^3-b^3}{a-b}$ and $\dfrac{a^3-b^3}{a^2+ab+b^2}$.
85. Suggestions for factoring.
86. Miscellaneous exercises in factoring.

CHAPTER XIII

SOLUTION OF EQUATIONS BY FACTORING

87. Solution of Equations by Factoring.

$3 \cdot 0 = ?$ $a \cdot 0 = ?$ $0 \cdot 5 = ?$ $0 \cdot b = ?$

What is the product of any number and 0 (zero)?

What is necessary in order that the product of two or more factors may be zero?

If the product of two or more factors is zero, at least one of the factors is zero, and if one of the factors of the product is zero, the product is zero.

This principle may be used in the solution of certain equations.

ILLUSTRATIVE EXERCISES

1. Solve: $x^2 - 6x + 5 = 0$ (1)

Solution:

Factoring the first member,

$$(x - 5)(x - 1) = 0 \qquad (2)$$

The solutions of (2) are the solutions of (1).

To solve (2), a number must be found which, when substituted for x in (2), will reduce it to an identity. It is necessary, then, to find a value of x which will make the first member reduce to zero. But the first member of (2) is zero when, and only when, one of its factors is zero. x must therefore be a number that makes

$$x - 5 = 0 \qquad (3)$$

or,

$$x - 1 = 0 \qquad (4)$$

From (3) $x = 5$, and from (4), $x = 1$.

Either of these values substituted for x in the original equation satisfies the equation. Substituting 1,

$$1 - 6 + 5 = 0$$

and substituting 5,

$$25 - 30 + 5 = 0$$

2. Solve: $x^2 + 4x - 12 = 0.$ (1)

Solution:

Factoring the first member,

$$(x + 6)(x - 2) = 0 \qquad (2)$$

In order that $(x + 6)(x - 2)$ shall be equal to zero, it is necessary that

$$x + 6 = 0, \text{ whence } x = -6$$

or,

$$x - 2 = 0, \text{ whence } x = 2$$

Substituting -6 for x in the original equation,

$$36 - 24 - 12 = 0$$

Substituting 2 for x,

$$4 + 8 - 12 = 0$$

3. Solve: $x^3 - x^2 - 4x + 4 = 0.$ (1)

Solution:

Factoring the first member of (1),

$$x^2(x - 1) - 4(x - 1) = (x^2 - 4)(x - 1)$$
$$= (x + 2)(x - 2)(x - 1)$$

Hence,

$$(x + 2)(x - 2)(x - 1) = 0 \qquad (2)$$

This equation is satisfied if any one of the factors of the first member is equal to zero, that is, if

$$x + 2 = 0 \text{ or } x - 2 = 0 \text{ or } x - 1 = 0$$

Hence,

$$x = -2 \text{ or } x = 2 \text{ or } x = 1$$

Substituting -2 for x in the original equation,

$$-8 - 4 + 8 + 4 = 0$$

Substituting 2 for x,

$$8 - 4 - 8 + 4 = 0$$

Substituting 1 for x,

$$1 - 1 - 4 + 4 = 0$$

To solve an equation by factoring:

1. Transform the equation so that the right hand member is zero.

2. Factor the left hand member, and put each factor equal to zero.

3. Solve the equations so formed. Their solutions are the solutions of the given equation.

EXERCISES

Solve the following equations, and check results in each case:

1. $x^2 + 5x + 6 = 0$
2. $x^2 - 5x + 6 = 0$
3. $x^2 + x - 6 = 0$
4. $x^2 - x - 6 = 0$
5. $m^2 + 8m + 15 = 0$
6. $n^2 - 2n - 35 = 0$
7. $a^2 + 4a - 21 = 0$
8. $x^2 - 7x + 10 = 0$
9. $z^2 + z - 30 = 0$
10. $y^2 - 2y - 24 = 0$
11. $y^2 - 5y - 24 = 0$
12. $c^2 + 4c - 32 = 0$
13. $c^2 + 12c + 27 = 0$
14. $c^2 - 15c + 50 = 0$
15. $a^2 - a - 110 = 0$
16. $x^2 + 5x - 84 = 0$
17. $x^2 - 18x = 40$
18. $y^2 - 6y = 55$
19. $y^2 - 14y = + 83$
20. $y^2 - 10y = 96$
21. $s^2 - 7s - 98 = 0$
22. $s^2 - 22s + 120 = 0$
23. $s^2 + 3s - 108 = 0$
24. $c^2 - 12c - 45 = 0$
25. $x^2 - 7x - 44 = 0$
26. $x^2 - 3x = 0$
    ($x = 0$ or $x = 3$)
27. $x^2 - 7x = 0$
28. $x^2 + 4x = 0$
29. $x^3 - 3x^2 - 88x = 0$
30. $x^2 + 9x - 70 = 0$
31. $2x^2 = 3 - 5x$
32. $3x^2 - 5x - 2 = 0$
33. $2x^2 + x = 6$
34. $2x^2 = 3x + 5$
35. $2x^2 + 7x + 6 = 0$
36. $5x^2 + 6 = 13x$
37. $3y^2 + 4y = 4$
38. $2y^2 - 7y - 15 = 0$
39. $2a^2 - 9a = 4a - 21$
40. $5a^2 - 11a = 10 - a^2$
41. $10a^2 - 9a = 9$
42. $15a^2 - 8 = 14a$
43. $2y^3 - 3y^2 - 8y + 12 = 0$

44. $3x^3 + 4x^2 - 3x - 4 = 0$
45. $5x^3 - 3x^2 - 5x + 3 = 0$
46. $x^3 - 4x^2 - 9x + 36 = 0$
47. $x^3 - 3x^2 - 4x + 12 = 0$
48. $x^2 - (a + 3)x + 3a = 0$
49. $x^2 - (a + b)x + ab = 0$
50. $x^2 + (c + 5)x + 5c = 0$

How many of the above problems can you solve by inspection?

In each of the following eight equations, find one root:

51. $x^3 - 1 = 0$
52. $x^3 + 1 = 0$
53. $x^3 - 8 = 0$
54. $x^3 - 4x^2 + 2x - 8 = 0$
55. $x^3 - 2x^2 - 3x + 6 = 0$
56. $2y^3 - 4y^2 + 3y - 6 = 0$
57. $3y^3 - 5y^2 + 6y - 10 = 0$
58. $x^3 - 27a^3 = 0$

59. The product of two numbers is 56 and their difference is 10. Find the numbers.

Solution:

Let x = the smaller number.

Then $x + 10$ = the larger number.
and $x(x + 10)$ is their product.

But 56 is the product of the two numbers. Hence,

$$x(x + 10) = 56 \qquad (1)$$

Removing parenthesis,

$$x^2 + 10x = 56 \qquad (2)$$

Subtracting 56 from each member of (2),

$$x^2 + 10x - 56 = 0 \qquad (3)$$

Factoring the first member of (3),

$$(x + 14)(x - 4) = 0 \qquad (4)$$

Hence, $x = -14$, or $x = 4$.

The smaller number, therefore, is -14 or 4. If the smaller is -14, the larger is $-14+10$, or -4. If the smaller number is 4, the larger is $4+10$, or 14. Since $(-14)\cdot(-4)=56$ and $14\cdot 4=56$, these results satisfy the conditions of the problem and are correct.

60. The product of two consecutive integers is 132. What are the integers?

61. The difference between two numbers is 6 and their product is 187. What are the numbers?

62. The area of a rectangular piece of land is 720 square rods. The length is 6 rods greater than the width. Find the dimensions of the field.

63. Find two consecutive integers whose product is 240.

64. Find a number which is 42 less than its square.

65. The difference between two numbers is 7 and the sum of their squares is 169. What are the numbers?

66. The sum of the two digits of a number is 7 and the product of the digits is 12. What two numbers answer these conditions?

67. Find two consecutive integers such that their product exceeds 8 times the smaller number by 8.

68. Find a number whose square exceeds 10 times the number by 39.

69. Divide 37 into two parts whose product is 300.

70. A field contains 6 acres. It is 2 rods longer than it is wide. What are its dimensions?

71. A farm sold for $7200. The number of dollars per acre was 10 less than the number of acres. How many acres were there in the farm?

CHAPTER XIV

HIGHEST COMMON FACTOR AND LOWEST COMMON MULTIPLE

89. An expression that is a factor of two or more expressions is called a **common factor** of these expressions. Thus, 3 is a common factor of 9 and 15. x is a common factor of $4ax^2$ and x^2y. $a + 2$ is a common factor of $a^2 - 4$ and $a^2 + 4a + 4$.

The **highest common factor** of two or more expressions is the product of all the common prime factors. For example, the common prime factors of $5a^2x$ and $15xy$ are 5 and x. Hence, $5x$ is the highest common factor of $5a^2x$ and $15xy$.

In the expressions $5a^2bc$, $15ab^2c$, and $20abc^2$, 5, a, b, and c are the common prime factors. Therefore, $5abc$ is the highest common factor.

Highest common factor is often abbreviated to H. C. F.

As indicated in the definition, to find the H. C. F. of two or more expressions,

Find the prime factors of each expression and write the product of the factors common to all, using each common factor the least number of times it occurs in any one of the given expressions.

ILLUSTRATIVE EXERCISES

1. Find the H. C. F. of 36, 24, and 48.

Solution:

$$36 = 2^2 \cdot 3^2$$
$$24 = 2^3 \cdot 3$$
$$48 = 2^4 \cdot 3$$

The common factors are 2^2 and 3, and the H. C. F. is $2^2 \cdot 3 = 12$.

2. Find the H. C. F. of $4a^2b$, $12a^2b^2c$, and $6ab^3c^3$.

Solution:

$$\begin{aligned} 4a^2b &= 2^2\cdot a^2\cdot b \\ 12a^2b^2c &= 2^2\cdot 3\cdot a^2\cdot b^2\cdot c \\ 6ab^3c^3 &= 2\cdot 3\cdot a\cdot b^3\cdot c^3 \\ \text{H. C. F.} &= 2\cdot a\cdot b = 2ab \end{aligned}$$

EXERCISES

Find the H. C. F. of the following, by inspection when possible:

1. a^2b, ab^2 a

2. ax^2, bx^2, cx^2

3. $3x^3y$, $6x^3$ 3

4. $10a^2$, $15a^3$, $20a$

5. $4a^3b^2$, $6ab^2$ 2a

6. $x^2y^2z^2$, $a^2x^3y^2$, $b^3x^2y^4z$ y^2

7. $2a^2c^3x$, $9a^3c^2n$, $6a^3c$

8. $18xyz$, $24yzw$, $33yzw^2$

9. $3a^2cd$, $27ac^3d^3$, $9a^3c^2d^2$

10. $4xyz^2$, $12x^3y^4z^3$, $20x^2y^2z^2$

11. $15m^4np^3$, $10m^2n^3p^5$, $25m^4n^2p$ 5

12. $14x^2y$, $21x^3y$, $42x^2y^2z$, $35xy^5z^2$ 7

13. $4a(a+b)$, $12x(a+b)$, $2g(a+b)^2$

14. $5(x+y)^2(x-y)$, $25a(x+y)(x-y)^2$, $15b(x+y)(x-y)$

15. $7a^2x^2y^3$, $14d^2x^3y^2$, $21c^2x^3y^3$ 7

16. $12m^3n^2$, $18mn^5$, $6m^5n^3$

17. $8a^3x^2$, $12a^3x^2z$, $20ax^4z^2$

18. $14m^2x^5$, $22am^3x^3$, $26cm^2x^4$ 2

19. $a^2b + ab^2$, a^3b

Solution:

$$\begin{aligned} a^2b + ab^2 &= ab(a+b) \\ a^3b &= a^3\cdot b \end{aligned}$$

The common factors are a and b. Hence,

$$\text{H. C. F.} = a\cdot b = ab$$

20. $3a^2c + 3ac^2$, $a^2 + 2ac + c^2$ a^2 c^2

21. $a^2 - 2ax + x^2,\ a^2 - x^2$ 2 a²
22. $a^2 - 3a + 2,\ a^2 - 2a + 1$ 3 a²
23. $x^2 - 5x + 6,\ x^2 - 3x$ 6 y²
24. $x^2 - 3x - 10,\ x^2 - 4$
25. $x^2 - 5x - 24,\ x^2 + 4x + 3$
26. $x^3 + 4x^2 - 77x,\ x^2 - 7x$ 77
27. $(a + b)^2 - 3(a + b) - 18,\ (a + b + 3)^2$
28. $a^2 - 2ab + b^2 - c^2,\ a^2 + 2ac + c^2 - b^2$
29. $x^2 + 6x + 9,\ x^3 + 27$ 3 y
30. $x^2 - y^2,\ x^3 - y^3$
31. $6x^2 + 7x - 3,\ 2x^2 - 5x - 12$
32. $m^4 - n^4,\ m^3 - n^3,\ m^2 - n^2$
33. $x^2 - 4,\ x^2 - x - 2,\ x^3 - 8$
34. $c^3 - 27,\ c^2 - 6c + 9,\ c^2 + 2c - 15$
35. $y^2 - 9,\ y^2 + 12y + 27,\ y^2 + 11y + 24$
36. $6x^2 - 23x - 35,\ x^2 - 7x + 10,\ x^2 - x - 20$
37. $9x^2 - 4,\ 27x^3 - 8,\ 12x^2 + x - 6$
38. $25a^2 - 70a + 49,\ 125a^3 - 343,\ 10a^2 + a - 21$
39. $a^2x - x^3,\ ax + x^2,\ a^3x + x^4$
40. $x^3 + 3x^2 - 10x,\ 2x^3 + 11x^2 + 5x$

If one of two expressions can be factored by inspection and the other can not, the H. C. F. can be found. For example,

41. Find the H. C. F. of $2x^2 - x - 6$ and $2x^3 - x^2 + 4x + 15$.

Solution:

Factoring the first expression,

$$\begin{aligned} 2x^2 - x - 6 &= 2x^2 - 4x + 3x - 6 \\ &= (x - 2)(2x + 3) \end{aligned}$$

It is evident that $x - 2$ is not a factor of $2x^3 - x^2 + 4x + 15$, since the last term, 2, is not an integral factor of 15. Hence the only possible common factor is $2x + 3$. Dividing $2x^3 - x^2 + 4x + 15$ by $2x + 3$, we find it is contained $x^2 - 2x + 5$ times. Since $2x + 3$ is a factor of both expressions, and is the only factor common to both of them, it is the H. C. F.

Find the H. C. F. of the following:

42. $x^2 - 5x + 6,\ x^3 - x^2 - x - 15$

43. $x^3 + 2x^2 - 15x,\ x^3 - 4x^2 + 6x - 9$

44. $x^2 + 7x + 10,\ x^3 + 6x^2 + 14x + 45$

45. $x^2 - 6x + 9,\ x^3 - 11x^2 + 39x - 45$

46. $3x^2 - 7x + 4,\ 6x^3 - 23x^2 + 11x + 12$

47. $x^3 + x^2 - 10x + 8,\ x^3 - x^2 - 4x + 4$

90. Lowest Common Multiple.

A **common multiple** of two or more expressions is an expression of which each of the given expressions is a factor. Thus, 24 is a common multiple of 3, 4, 6, and 12. $a^2 - x^2$ is a common multiple of $a + x$ and $a - x$.

The **lowest common multiple** of two or more expressions is the product of their different prime factors, each factor being used the greatest number of times that it occurs in any one of the given expressions.

Lowest common multiple is often abbreviated L. C. M.

ILLUSTRATIVE EXERCISES

1. Find the L. C. M. of 12, 36, and 30.

Solution:

$$12 = 2^2 \cdot 3$$
$$36 = 2^2 \cdot 3^2$$
$$30 = 2 \cdot 3 \cdot 5$$

The different prime factors of the given numbers are 2, 3, and 5. The greatest number of times that 2 occurs in any one number is twice. The greatest number of times that 3 occurs in any one number is twice, and 5 occurs only once in any one of the numbers. Hence, the L. C. M. of the given numbers is the product obtained by using 2 twice, 3 twice, and 5 once as factors. That is, the L. C. M. of 12, 36, and 30 is

$$2^2 \cdot 3^2 \cdot 5 = 180$$

2. Find the L. C. M. of $14ab^4c^2$, $28a^2b^3cd^2$, and $2b^3c$.

Solution:

Factoring the given expressions, we have

$$\begin{aligned} 14ab^4c^2 &= 2 \cdot 7 \cdot a \cdot b^4 \cdot c^2 \\ 28a^2b^3cd^2 &= 2^2 \cdot 7 \cdot a^2 \cdot b^3 \cdot c \cdot d^2 \\ 2b^3c &= 2 \cdot b^3 \cdot c \end{aligned}$$

The different prime factors of the given expressions are 2, 7, a, b, c, and d. The greatest number of times that each occurs in any one of the given expressions is as follows: 2 occurs twice, 7 once, a twice, b 4 times, c twice, and d twice. Hence, the L. C. M. is

$$2^2 \cdot 7 \cdot a^2 \cdot b^4 \cdot c^2 \cdot d^2 = 28a^2b^4c^2d^2$$

EXERCISES

Find the L. C. M. of the following, by inspection as far as possible:

1. $6a^2b$, $9a^3b^2$ a^2b^2

2. $8x^5y$, $6x^3y^2z$

3. $12m^2n$, $20mn^2$

4. $2ab$, $3cx$, $4y$

5. $2a^2$, a^2y, $8a^2b^2$ $8a^2b^2y$

6. $15amr$, $45m^2r^3$, $5m^3n^3$

7. $9abm^2$, $12a^2mn$, $18bn^2$ 36

8. $9x^2y^2$, $24xy^2z^3$, $36x^2z$ $72x^2$

9. $27a^4$, $9a$, $18a^2$ $54a^4$

10. $3a^2x$, $12ax^2$, $18a^3x$, $4ax^3$

11. $36x(m+n)(m-n)$, $(m+n)^2$, $12x^2(m+n)^2$

$36x^2(m+n)^2(m-n)$

12. $x^2 + ax$, $x^2 + 2ax + a^2$

Solution:

$$x^2 + ax = x(a + x)$$
$$x^2 + 2ax + a^2 = (x + a)^2$$

The different factors are x and $x + a$. x occurs only once in any expression while $x + a$ occurs twice in one expression. Hence, the L. C. M. is

x

$$x(x + a)^2 = x^3 + 2ax^2 + a^2x$$

13. $a^2 - ax$, $a^2 + ax$ a(a-x) a(a+x)

14. $m^2 - 4$, $m^2 + 2m$

15. $a^2 - 5a + 6$, $a^2 + 5a - 14$

16. $y^2 - 16$, $y^2 - 9y + 20$

17. $x^2 + x + 1$, $x^3 - 1$

18. $a^2 - 4$, $a^2 + 4$, $a^4 - 16$

19. $3x^2 + 5x - 2$, $x^2 - 4$

20. $x^2 - 2x + 4$, $x^3 + 8$

21. $6x^2 + 13x - 28$, $12x^2 - 31x + 20$

22. $ax - ay - cx + cy$, $x^2 - 2xy + y^2$

23. $ma^2 + 3amx + 9mx^2$, $a^3 - x^3$, $a^2 - x^2$

24. $c^2 - 5c - 14$, $c^2 - 2c - 8$, $c^2 - 11c + 28$

25. $6m^2 + 23m - 4$, $2m^2 + 3m - 20$, $12m^2 - 32m + 5$

26. $27x^3 + 64$, $9x^2 - 16$, $9ax^2 - 12ax + 16a$

REVIEW EXERCISES

91. **1.** Factor:

(a) $a^2x^3 - 3ax^2 + 5ax$
(b) $27m^3 - 64n^3$
(c) $6x^3 - 9x^2 - 4x + 6$

Solve the following equations:

2. $x^2 - 3x - 70 = 0$

3. $x^2 + 5x - 66 = 0$

4. $x^2 + x - 56 = 0$

5. $x^2 - 2x - 63 = 0$

6. $x^2 - 14x + 33 = 0$

7. $2x^2 - x - 21 = 0$

8. $3x^2 + x - 10 = 0$

9. $10x^2 + x - 21 = 0$

10. What is the highest common factor of two or more numbers?

11. What is the lowest common multiple of two or more numbers?

Find the H. C. F. and L. C. M. of the following:

12. $9a^2y^3$, $12axy^2$, and $15a^3x^2y^2$

13. $ax^2 - 4a$, and $x^3 - 5x^2 + 6x$

14. $y^3 + 1$, and $2y^3 - y^2 - 3y$

15. $4x^2 + 12x + 9$, and $6x^2 + 19x + 15$

16. $x^2 + (a + b)x + ab$, and $x^3 + a^3$

17. If a is contained in x, 5 times and in y, 3 times, how many times is it contained in $x + y$? in $x - y$?

18. If a is contained in m, k times, and in n, h times, how many times is a contained in $m + n$? in $m - n$?

19. The difference between two numbers is 10 and their product is 144. What are the numbers?

20. Multiply $a^2 + ab + b^2$ by $a^2 - ab + b^2$.

21. What are the factors of $a^4 + a^2b^2 + b^4$? (See problem 20.)

SUMMARY

SECTION

89. Highest common factor. Problems.
90. Lowest common multiple. Problems.
91. Review exercises.

CHAPTER XV

FRACTIONS

92. Definition of Fraction. In algebra a fraction is an indicated division. The dividend is the numerator and the divisor the denominator. The numerator and denominator are the terms of the fraction.

93. Multiplication and Division of the Terms of a Fraction by the Same Number. In arithmetic both terms of a fraction may be multiplied or divided by the same number without changing the value of the fraction. We shall show that the same rule holds true in algebra.

Let $$\frac{a}{b} = x \qquad (1)$$

Since the dividend is equal to the product of the divisor and quotient,

$$a = bx \qquad (2)$$

Multiplying both terms of (2) by c (axiom IV),

$$ac = bcx \qquad (3)$$

Dividing both members of (3) by bc (axiom V),

$$\frac{ac}{bc} = x \qquad (4)$$

Since the x's in (1) and (4) are the same, by axiom I,

$$\frac{ac}{bc} = \frac{a}{b} \qquad (5)$$

Note. Equations 4 and 5 are not necessarily true if $c = 0$.

From (5) we have the following principle:

Both terms of a fraction may be multiplied or divided by the same number without changing the value of the fraction.

94. The Signs of a Fraction. Every fraction has three signs, *viz.*, the sign of the numerator, the sign of the denominator, and the sign of the whole fraction.

Thus, $\frac{a}{b}$ may be written $+\frac{+a}{+b}$. Here all the signs are positive. The fraction $\frac{-a}{+b}$ may be written $+\frac{-a}{+b}$. Here the sign of the numerator is negative and the other two signs are positive. When the sign of the whole fraction is not written, the plus sign is understood.

Since a fraction is an indicated division, the law of signs for division applies. By the law of signs,

$$\frac{4}{2}=\frac{-4}{-2}=-\frac{-4}{2}=-\frac{4}{-2}$$

Each of the above fractions is equal to 2. In the same way,

$$\frac{x}{y}=\frac{-x}{-y}=-\frac{-x}{y}=-\frac{x}{-y}$$

Of the three signs involved in any fraction, any two may be changed without altering the value of the fraction.

Thus, $\frac{5}{x-3}=\frac{-5}{3-x}=-\frac{-5}{x-3}=-\frac{5}{3-x}$

EXERCISES

Write each of the following fractions in three other ways:

1. $\frac{a}{2b}$	**5.** $\frac{3}{1-a}$	**9.** $\frac{4}{a^2-x^2}$
2. $-\frac{3}{4}$	**6.** $\frac{am}{b}$	**10.** $\frac{3rx}{2c+b}$
3. $\frac{2a}{c}$	**7.** $\frac{c}{2-x}$	**11.** $\frac{m^2-2}{1-m^2}$
4. $-\frac{a^2}{bx}$	**8.** $\frac{3x-2}{m+n}$	**12.** $\frac{8a+4b}{2mn}$

95. Reduction of Fractions to Lowest Terms.

A fraction is said to be in its *lowest terms* when the numerator and the denominator have no common factor.

To reduce a fraction to lowest terms:

Separate the numerator and the denominator into their prime factors and divide both terms by their common factors.

Reduce $\frac{64a^2x^3}{84a^3x^2}$ to lowest terms.

$$\frac{64a^2x^3}{84a^3x^2} = \frac{2^6\cdot a^2\cdot x^3}{2^2\cdot 3\cdot 7\cdot a^3\cdot x^2} = \frac{2^4\cdot x}{3\cdot 7\cdot a} = \frac{16x}{21a}$$

The numerator and the denominator have been divided by $2^2\cdot a^2\cdot x^2$.

The division of numerator and denominator by the common factors may be indicated as in arithmetic by cancelling these factors.

EXERCISES

Reduce the following fractions to lowest terms. Solve as many of the problems as possible mentally.

1. $\frac{48}{64}$ $\div 16 = 3/4$
2. $\frac{38}{57}$ $\div 19 = 2/3$
3. $\frac{ab}{ac}$ $\div a = b/c$
4. $\frac{72}{84}$ $\frac{6}{7}$
5. $\frac{56}{108}$ $= \frac{14}{27}$
6. $\frac{52}{91}$ $\frac{4}{7}$
7. $\frac{a^3}{a^2c}$
8. $\frac{3a^2b^3}{6a^3b^2}$ $= \frac{b}{2a}$
9. $\frac{18x^3y}{24y^3}$ $\frac{3x^3}{4y^2}$
10. $\frac{84cd^2}{72c^3d}$ $\frac{7d}{6c^2}$
11. $\frac{84c^2x^3}{60c^3x^2}$ $= \frac{7x}{5c}$
12. $\frac{a(a-b)}{x(a-b)}$ $= \frac{a}{x}$
13. $\frac{x(x-2)}{x^2-4}$ $\frac{x}{x+2}$
14. $\frac{a^2-b^2}{a^2+2ab+b^2}$
15. $\frac{5a(m-3)}{m^2-5m+6}$ $\frac{5a}{m-2}$
16. $\frac{3a(a+b)}{a^2+3ab+2b^2}$
17. $\frac{3x(x-y)}{2y(x^2-y^2)}$
18. $\frac{x^2-y^2}{x^3-y^3}$
19. $\frac{m^2-n^2}{m^6-n^6}$
20. $\frac{4a^2+12ay+9y^2}{8a^3+27y^3}$
21. $\frac{y^2-3y-40}{y^2-11y+24}$
22. $\frac{a^3+3a^2-18a}{a^3-5a^2-66a}$
23. $\frac{x^3-2x^2-6x+12}{x(x^2-9x+14)}$
24. $\frac{ax^2-4a^3-bx^2+4a^2b}{ax+a^2-bx-ab}$
25. $\frac{12m^2-5m-2}{8m^2-18m-5}$
26. $\frac{18a^2+13a-21}{8a^3x+27x}$

27. $$\frac{x^2 - 4x + 3}{1 - x^3} = -\frac{x^2 - 4x + 3}{x^3 - 1}$$
$$= -\frac{(x-1)(x-3)}{(x-1)(x^2+x+1)}$$
$$= -\frac{x-3}{x^2+x+1}$$

28. $\dfrac{x^2 - 4x + 4}{4 - x^2}$

29. $\dfrac{4a^2 - 20ax + 25x^2}{125x^3 - 8a^3}$

30. $\dfrac{12b^2 - 4b - 21}{27 - 8b^3}$

31. $\dfrac{16m^2y - 25y^3}{10y^3 + 7y^2m - 12ym^2}$

96. Reduction of Fractions to a Common Denominator.

ILLUSTRATIVE EXERCISES

1. Reduce $\frac{3}{4}$ to an equivalent fraction having 12 for a denominator.

Solution:

This may be done by multiplying the numerator and the denominator by 3, thus:

$$\frac{3}{4} = \frac{3 \cdot 3}{3 \cdot 4} = \frac{9}{12}$$

2. Reduce $\dfrac{a}{a-x}$ to a fraction whose denominator is $a^2 - x^2$.

Solution:

This may be done by multiplying the numerator and the denominator by $a + x$.

$$\frac{a}{a-x} = \frac{a(a+x)}{(a-x)(a+x)} = \frac{a^2+ax}{a^2-x^2}$$

3. Reduce $\frac{3x}{x^2 - a^2}$ and $\frac{2a}{x^2 - 4ax + 3a^2}$ to equivalent fractions having a common denominator.

Solution:

The L. C. M. of the denominators is $(x - a)(x + a)(x - 3a)$. To make the denominator of the first fraction $(x - a)(x + a)(x - 3a)$, its denominator must be multiplied by $x - 3a$. If the denominator is multiplied by $x - 3a$, the numerator must be multiplied by the same number, and

$$\frac{3x}{x^2 - a^2} = \frac{3x(x - 3a)}{(x - a)(x + a)(x - 3a)}$$

$$= \frac{3x^2 - 9ax}{(x - a)(x + a)(x - 3a)}$$

To make the denominator of the second fraction

$$(x - a)(x + a)(x - 3a)$$

its denominator must be multiplied by $x + a$. Multiplying the numerator and the denominator by $x + a$, we obtain

$$\frac{2a}{x^2 - 4ax + 3a^2} = \frac{2a(x + a)}{(x - a)(x + a)(x - 3a)}$$

$$= \frac{2ax + 2a^2}{(x - a)(x + a)(x - 3a)}$$

To reduce fractions to equivalent fractions having a common denominator, we have the following rule:

Find the L. C. M. of the denominators.

Multiply both terms of each fraction by those factors of the L. C. M. not contained in the denominator of the fraction.

EXERCISES

Reduce the following to equivalent fractions having a common denominator, performing the work mentally when possible:

1. $\frac{3}{4}, \frac{2}{9}, \frac{1}{12}$

2. $\frac{a}{2}, \frac{b}{4}, \frac{c}{2}$

3. $\frac{x}{3}, \frac{x}{5}, \frac{xy}{10}$

4. $\frac{3m}{4}, \frac{2n}{8}, \frac{mn}{12}$

5. $\frac{3}{a}, \frac{a}{3}$

6. $\frac{2}{3x}, \frac{3}{2y}, \frac{a}{6xy}$

7. $\frac{x}{mn}, \frac{2y}{m^2}, \frac{3x}{n^2}$

8. $\frac{4}{3xy}, \frac{3}{8x^2y^2}, \frac{5}{6x^2}$

9. $\frac{2}{a^3b^2m}, \frac{3a}{4a^2m^2}, \frac{x}{m^2n^2}$

10. $\frac{a+b}{x}, \frac{a-b}{x^2}$

11. $\frac{2}{a-b}, \frac{3}{a+b}$

12. $\frac{5}{x-2}, \frac{7}{x+2}$

13. $\frac{a}{m-3}, \frac{b}{m+3}, \frac{c}{m^2-9}$

14. $\frac{2}{x+2}, \frac{3}{x+5}, \frac{4}{x^2+7x+10}$

15. $\frac{5}{m-2}, \frac{2}{m^3-8}, \frac{6}{m^2+2m+4}$

16. $\frac{2ab+b}{mn}, \frac{2m+n}{ab}, \frac{b}{am}$

17. $\frac{x}{a+b}, \frac{y}{a^2+ab}$

18. $\frac{a+b}{a-b}, \frac{a-b}{a+b}$

19. $\frac{a}{a+b}, \frac{b}{a-b}, \frac{a^2b^2}{a^2-b^2}$

20. $\frac{3x}{x-2}, \frac{5x}{x+2}, \frac{4x+3}{x^2-4}$

21. $\dfrac{2}{x^2-5x+6}, \dfrac{3}{x^2-x-2}$

22. $\dfrac{a}{m^2-mn}, \dfrac{m+n}{m^3-n^3}$

23. $\dfrac{3}{x}, \dfrac{x+2}{9x^2-1}, \dfrac{3x-1}{3x^2+7x+2}$

24. $\dfrac{2x}{x+2}, \dfrac{3}{2-x}, \dfrac{x-1}{x^2-4}$

Solution:

The common denominator of these fractions is x^2-4 or $4-x^2$. In reducing the fractions to equivalent fractions having a common denominator it is convenient to change the signs of both terms of the second fraction and use x^2-4 as the common denominator. Then,

$$\frac{2x}{x+2}=\frac{2x(x-2)}{(x+2)(x-2)}=\frac{2x^2-4x}{x^2-4}$$

and

$$\frac{3}{2-x}=\frac{-3}{x-2}=\frac{-3(x+2)}{(x-2)(x+2)}=\frac{-3x-6}{x^2-4}$$

and the third fraction remains unchanged, $\dfrac{x-1}{x^2-4}$.

25. $\dfrac{2}{a+x}, \dfrac{3}{x-a}, \dfrac{4}{a^2-x^2}$

26. $\dfrac{a}{a^2-2ax+x^2}, \dfrac{x}{x^2-a^2}$

27. $\dfrac{2a-b}{b-2}, \dfrac{2b.}{4-b^2}$

28. $\dfrac{x+3}{1-9x^2}, \dfrac{2x+3}{3x^2+5x-2}$

29. $\dfrac{4m}{m^2+2m+4}, \dfrac{m^2-2}{8-m^3}$

30. $\dfrac{3}{x^4-y^4}, \dfrac{4}{x^2+y^2}, \dfrac{5}{y^2-x^2}$

97. Addition and Subtraction of Fractions.

Fractions are added and subtracted in algebra in the same manner as in arithmetic.

ILLUSTRATIVE EXERCISES

1. Add $\frac{2}{9}$, $\frac{4}{9}$, and $\frac{7}{9}$.

Solution: $\frac{2}{9}+\frac{4}{9}+\frac{7}{9}=\dfrac{2+4+7}{9}=\frac{13}{9}$

2. Add $\dfrac{2a}{4x^2}$, $\dfrac{3m}{4x^2}$, and $\dfrac{c}{4x^2}$.

Solution: $\dfrac{2a}{4x^2}+\dfrac{3m}{4x^2}+\dfrac{c}{4x^2}=\dfrac{2a+3m+c}{4x^2}$

3. Subtract $\dfrac{6ab}{x+6}$ from $\dfrac{4m^2}{x+6}$.

Solution: $\dfrac{4m^2}{x+6}-\dfrac{6ab}{x+6}=\dfrac{4m^2-6ab}{x+6}$

We check by letting $m=1$, $a=1$, $b=1$, and $x=1$.

$$\frac{4\cdot 1}{1+6}-\frac{6\cdot 1\cdot 1}{1+6}=\frac{4}{7}-\frac{6}{7}=-\frac{2}{7}$$

$$\frac{4\cdot 1-6\cdot 1\cdot 1}{1+6}=-\frac{2}{7}$$

Since $-\frac{2}{7}=-\frac{2}{7}$, the result checks.

CAUTION. Care must be taken in checking not to give values to the letters that will make a denominator equal to zero.

Fractions having a common denominator may be added or subtracted by adding or subtracting the numerators and writing the result over the common denominator.

EXERCISES

Perform the following additions and subtractions. The first fifteen or more problems should be worked by inspection. Reduce the results to lowest terms.

1. $\frac{4}{13}+\frac{5}{13}+\frac{3}{13}$

2. $\frac{2}{a}+\frac{3}{a}+\frac{7}{a}$

3. $\frac{8}{15}+\frac{8}{15}-\frac{5}{15}$

4. $\frac{5}{y}-\frac{12}{y}+\frac{6}{y}$

5. $\frac{a}{2}+\frac{b}{2}+\frac{c}{2}$

6. $\frac{x}{m}+\frac{5}{m}+\frac{y}{m}$

7. $\frac{a}{xy}-\frac{x}{xy}$

8. $\frac{6x}{m^2}+\frac{3x}{m^2}-\frac{12x}{m^2}$

9. $\frac{a}{a^2-b^2}-\frac{b}{a^2-b^2}$

10. $\frac{a+b}{c}+\frac{a-b}{c}$

11. $\frac{7}{x+y}+\frac{10}{x+y}-\frac{12}{x+y}$

12. $\frac{x^2}{x^2-4}-\frac{6x}{x^2-4}+\frac{8}{x^2-4}$

13. $\frac{a}{a^2-9}+\frac{3}{a^2-9}$

14. $\frac{x^2}{x^3+y^3}-\frac{xy}{x^3+y^3}+\frac{y^2}{x^3+y^3}$

15. $\frac{x}{x^2-y^2}-\frac{y}{x^2-y^2}+\frac{2y}{x^2-y^2}$

16. $\frac{m-n}{a+2b}-\frac{m+n}{a+2b}$

$$=\frac{m-n-(m+n)}{a+2b}$$

$$=\frac{m-n-m-n}{a+2b}=\frac{-2n}{a+2b}$$

17. $\frac{3a+2b}{x+y}-\frac{2b-3a}{x+y}$

18. $\frac{x-4}{2xy}-\frac{x-7}{2xy}-\frac{2-x}{2xy}$

19. $\dfrac{6x^2}{4x^2 - 12x + 9} - \dfrac{11x}{4x^2 - 12x + 9} + \dfrac{3}{4x^2 - 12x + 9}$

20. $\dfrac{m^2}{m^3 - 8} - \dfrac{2m - 3}{8 - m^3} + \dfrac{7}{m^3 - 8}$

Hint. Write the second fraction $+\dfrac{2m - 3}{m^3 - 8}$.

21. $\dfrac{a^2}{a^2 - b^2} - \dfrac{ab}{a^2 - b^2} + \dfrac{b^2}{a^2 - b^2} - \dfrac{ab}{a^2 - b^2}$

ILLUSTRATIVE EXERCISES

1. Add $\frac{3}{14}$, $\frac{5}{21}$, and $\frac{11}{28}$.

Solution:

The L. C. M. of the denominators is 84. Using this as a common denominator, we have

$$\frac{3}{14} + \frac{5}{21} + \frac{11}{28} = \frac{18}{84} + \frac{20}{84} + \frac{33}{84}$$
$$= \frac{18 + 20 + 33}{84} = \frac{71}{84}$$

Another denominator might have been used, such as 168 or 252, but it is usually advisable to use the L. C. M. of the denominators.

2. Combine into one fraction $\dfrac{3}{x - 1}$, $\dfrac{x + 3}{x^2 - 4}$, and $-\dfrac{2x - 1}{x^2 - 3x + 2}$.

The L. C. M. of the denominators is $(x - 1)(x - 2)(x + 2)$.

ellie and I sat in room no 2. Oct 1, 1927 Ha Ha.

$$\frac{3}{x-1}+\frac{x+3}{x^2-4}-\frac{2x-1}{x^2-3x+2}=\frac{3(x-2)(x+2)}{(x-1)(x+2)(x-2)}$$

$$+\frac{(x+3)(x-1)}{(x-1)(x+2)(x-2)}-\frac{(2x-1)(x+2)}{(x-1)(x+2)(x-2)} \qquad (1)$$

$$=\frac{3(x-2)(x+2)+(x+3)(x-1)-(2x-1)(x+2)}{(x-1)(x+2)(x-2)} \qquad (2)$$

$$=\frac{3x^2-12+(x^2+2x-3)-(2x^2+3x-2)}{(x-1)(x+2)(x-2)} \qquad (3)$$

$$=\frac{3x^2-12+x^2+2x-3-2x^2-3x+2}{(x-1)(x+2)(x-2)} \qquad (4)$$

$$=\frac{2x^2-x-13}{(x-1)(x+2)(x-2)} \qquad (5)$$

$$=\frac{2x^2-x-13}{x^3-x^2-4x+4} \qquad (6)$$

In most cases the last step may be omitted. In any case it is better to keep the common denominator in the factored form until the addition or subtraction is completed, for in this form it is easier to see by what each numerator and denominator must be multiplied.

We may check by letting $x = 3$.

$$\frac{3}{3-1}+\frac{3+3}{9-4}-\frac{6-1}{9-9+2}=\tfrac{3}{2}+\tfrac{6}{5}-\tfrac{5}{2}=\tfrac{1}{5}$$

$$\frac{2x^2-x-13}{x^3-x^2-4x+4}=\frac{18-3-13}{27-9-12+4}=\tfrac{2}{10}=\tfrac{1}{5}$$

In this example it is impossible to check by letting $x = 1$ or $x = 2$, for, in either case, some of the denominators would be zero.

In step 3 each numerator, after the first, is enclosed in a parenthesis preceded by the sign of the fraction. It is advisable to follow this plan until one is accustomed to the work.

To add or subtract fractions:

Reduce them to equivalent fractions having a common denominator, add or subtract each numerator according to the sign before the fraction, and write the result over the common denominator.

An integral expression may be regarded as a fraction whose denominator is 1. Thus, x may be written $\frac{x}{1}$, $a+b=\frac{a+b}{1}$, etc.

3. Combine $x+\frac{3}{a}-\frac{3y}{2}$.

The common denominator is $2a$, and we have

$$x+\frac{3}{a}-\frac{3y}{2}=\frac{2ax}{2a}+\frac{6}{2a}-\frac{3ay}{2a}=\frac{2ax+6-3ay}{2a}$$

Check by letting $x=1$, $a=1$, and $y=1$.

$$1+\tfrac{3}{1}-\tfrac{3}{2}=\tfrac{2}{2}+\tfrac{6}{2}-\tfrac{3}{2}=\tfrac{5}{2}$$

$$\frac{2\cdot 1\cdot 1+6-3\cdot 1\cdot 1}{2\cdot 1}=\frac{2+6-3}{2}=\tfrac{5}{2}$$

EXERCISES

Perform the following additions and subtractions. The first twenty or more problems should be worked by inspection.

1. $\frac{2}{5}+\frac{3}{10}+\frac{1}{2}$

2. $\frac{a}{2}+\frac{a}{4}$

Began

3. $\frac{x}{3} - \frac{x}{6} + \frac{x}{2}$

4. $\frac{5}{8} + \frac{1}{2} - \frac{3}{16}$

5. $\frac{a}{2} + \frac{2a}{4} + \frac{5a}{10}$

6. $\frac{4x}{3} - \frac{x}{6} - \frac{7x}{12}$

7. $\frac{3}{a} - \frac{3}{2a}$

8. $\frac{3}{x} - \frac{2}{2x} + \frac{x}{3}$

9. $\frac{a}{b} + \frac{b}{a}$

10. $\frac{a}{b} - \frac{b}{a} + \frac{2}{ab}$

11. $3 + \frac{1}{x}$

12. $2 + \frac{x-1}{x}$

13. $c + \frac{3}{2}$

14. $x + \frac{1}{a}$

15. $1 + \frac{a}{b}$

16. $\frac{1}{3} + \frac{3}{7} + \frac{8}{21}$

17. $\frac{x}{3} + \frac{2x}{5}$

53
24
24

18. $\frac{m}{2} + \frac{2m}{3} - \frac{3m}{4}$

19. $\frac{7y}{12} + \frac{3y}{10} - \frac{2y}{5}$

20. $a + \frac{3}{a-x}$

21. $x - 3 + \frac{2}{x+3}$

22. $\frac{m+1}{5} + \frac{m-1}{3}$

23. $\frac{2a-3}{15} + \frac{a+2}{10} - \frac{4a}{2}$

24. $1 + \frac{x-2}{4} - \frac{x}{3}$

25. $\frac{x}{2y}-\frac{5x}{4y}+\frac{3x}{6y}$

26. $\frac{1}{ab}+\frac{1}{ac}-\frac{1}{bc}$

27. $\frac{2x-3a}{4a}-\frac{2x-6b}{5b}$

28. $\frac{x-1}{3}+\frac{x+1}{2}+\frac{x-2}{5}$

29. $x^2-x+1-\frac{x^3}{x+1}$

30. $\frac{1}{a+2x}+\frac{3}{a-2x}$

31. $\frac{a+b}{a-b}+\frac{a-b}{a+b}$

32. $\frac{m+n}{3m}-\frac{m-n}{4n}+\frac{m-2n}{12mn}$

33. $\frac{1}{x-2}-\frac{2}{x+2}+\frac{3}{x^2-4}$

34. $\frac{m}{m-n}+\frac{2n}{m+n}-\frac{mn}{m^2-n^2}$

35. $\frac{a}{a+2}+\frac{2a}{3a-4}$

36. $\frac{4xy}{z}+\frac{3yz}{x}-\frac{7xz}{y}$

37. $a^2+a+1+\frac{a^3}{a-1}$

38. $m^2-n^2-\frac{m^2n}{m+n}$

39. $\frac{m-3}{x+2}-\frac{2m+4}{x-2}$

40. $a-b+c-\frac{a^2}{2ab}$

41. $\frac{x-y}{x+y}-\frac{x+y}{x-y}$

42. $\frac{4}{x+4}+\frac{3}{x+3}$

43. $\frac{a}{c}+\frac{b}{a}-\frac{c}{b}$

44. $\frac{c-d}{c+d}-\frac{c+d}{c-d}$

45. $\frac{6ab-3c}{4b}-2a$

46. $a^2-c^2-\frac{2a^2c+c^2}{2a-1}$

47. $x^2 - xy + y^2 - \dfrac{x^3 - y^3}{x + y}$

48. $\dfrac{3a - 5}{14a^2 - 13a - 12} - \dfrac{4a - 7}{21a^2 + 26a + 8}$

49. $\dfrac{ay}{a^2 - ay} - \dfrac{ax}{a^2 - ax} + \dfrac{xy}{x^2 - xy}$

50. $\dfrac{x^2 - xy + y^2}{x - y} - \dfrac{x^2 + xy + y^2}{x + y}$

51. $\dfrac{3}{a - b} + \dfrac{2}{a + b} + \dfrac{1}{b^2 - a^2}$

Hint. Write the last fraction $-\dfrac{1}{a^2 - b^2}$.

52. $\dfrac{2}{a - x} + \dfrac{3}{x + a} - \dfrac{4}{x^2 - a^2}$

53. $\dfrac{3}{x^4 - y^4} - \dfrac{4}{y^2 - x^2} + \dfrac{5}{x^2 + y^2}$

54. $\dfrac{3}{x^2 - 9} - \dfrac{2}{3 - x} - \dfrac{1}{3 + x}$

55. $\dfrac{2}{a - 1} - \dfrac{3}{(1 - a)^2} + \dfrac{1}{1 - a^2}$

56. $x - y - \dfrac{x^2 + y^2}{x + y}$

57. $2 + \dfrac{a + b}{a - b} - (a - b)$

58. $\dfrac{1}{a^2 + 2a - 8} - \dfrac{3}{3a - 6}$

59. $\dfrac{2x - 7}{3x} - \dfrac{x^2 + 5}{6} - \dfrac{4}{x^2}$

60. $\dfrac{2a + b}{b + 2} - \dfrac{2b}{4 - b^2}$

98. Multiplication of a Fraction by an Integral Expression. Since a fraction is an indicated division, multiplying the numerator (dividend) or dividing the denominator (divisor) multiplies the fraction. Thus, 5 is contained in 30 twice as many times as it is contained in 15. That is,

$$\frac{15}{5}\cdot 2 = \frac{30}{5} = \frac{15\cdot 2}{5}$$

The fraction $\frac{15}{5}$ is here multiplied by 2 by multiplying its numerator by 2. 3 is contained in 36 twice as many times as 6 is contained in 36. That is,

$$\frac{36}{6}\cdot 2 = \frac{36}{3} = \frac{36}{6 \div 2}$$

The fraction $\frac{36}{6}$ is here multiplied by 2 by dividing its denominator by 2.

k is contained in a, c times as many times as ck is contained in a. That is,

$$\frac{a}{ck}\cdot c = \frac{a}{k}$$

To multiply a fraction by an integral expression:

Multiply its numerator or divide its denominator by that expression.

ILLUSTRATIVE EXERCISES

1. Multiply $\frac{3}{8}$ by 4.

$$\frac{3}{\underset{2}{\cancel{8}}}\cdot \cancel{4} = \frac{3}{2}$$

Here we divided the denominator by 4.

2. Multiply $\frac{5}{7}$ by 3.

$$\frac{5}{7}\cdot 3 = \frac{15}{7}$$

Here we multiplied the numerator by 3.

3. Multiply $\dfrac{5}{a^2 - x^2}$ by $a + x$.

$$\frac{5}{a^2 - x^2} \cdot \frac{a + x}{1} = \frac{5}{(a - x)\cancel{(a + x)}} \cdot \frac{\cancel{a + x}}{1} = \frac{5}{a - x}$$

EXERCISES

Multiply, by inspection when possible:

1. $\frac{3}{15}$ by 5

2. $\frac{7}{5}$ by 3

3. $\dfrac{cd}{ab}$ by b

4. $\dfrac{a}{b}$ by c

5. $\dfrac{x^2y^3}{z^3}$ by z^2

6. $\dfrac{x^3y}{ab^3}$ by z^4

7. $\dfrac{x}{ab^3}$ by ab

8. $\dfrac{4pq}{x^3}$ by $5x$

Hint. Multiply the numerator by 5 and divide the denominator by x.

9. $\dfrac{3}{16a^2x}$ by $8am$

10. $\dfrac{5xy}{42m^3n^3}$ by $7mn$

11. $\dfrac{2}{a - x}$ by a

12. $\dfrac{2a}{x^2 - y^2}$ by $x - y$

13. $\dfrac{x}{x^2 - 9}$ by $x + 3$

14. $\dfrac{x + 4}{x^2 - 4}$ by $x - 2$

15. $\dfrac{3}{a - b}$ by $a + b$

16. $\dfrac{4m}{m^2 - 2m + 1}$ by $m - 1$

17. $\dfrac{a + 4}{a^2 - a - 6}$ by $a^2 + 2a$

18. $\dfrac{2a - 1}{25m^2 - 9}$ by $5m + 3$

19. $\dfrac{a + 1}{a + 3}$ by $a + 2$

20. $\dfrac{a + y}{4y^2}$ by $ay - y^2$

21. $\dfrac{2ay}{x^3 - y^3}$ by $x^2 - y^2$

99. Division of a Fraction by an Integral Expression. **A fraction is divided by an integral expression when its numerator is divided or its denominator multiplied by that expression.**

Thus, 5 is contained in 40, 4 times as many times as it is contained in 10. That is,

$$\frac{40}{5} = 4 \cdot \frac{10}{5}, \text{ or } \frac{40}{5} \div 4 = \frac{10}{5} \qquad (1)$$

We have also $\frac{40}{5} = 4 \cdot \frac{40}{20}$, since each equals 8. Hence,

$$\frac{40}{5} \div 4 = \frac{40}{20}. \qquad (2)$$

b is contained in ax, x times as many times as it is contained in a. That is,

$$\frac{ax}{b} = x \cdot \frac{a}{b}, \text{ or } \frac{ax}{b} \div x = \frac{a}{b} \qquad (1)$$

Since $\frac{ax}{bx} = \frac{a}{b}$, we have by axiom I

$$\frac{ax}{b} \div x = \frac{ax}{bx} \qquad (2)$$

In (1) we divided the numerator by the divisor; in (2) we multiplied the denominator by the divisor.

ILLUSTRATIVE EXERCISES

1. Divide $\frac{8}{9}$ by 4.

Solution:

$$\frac{8}{9} \div 4 = \frac{8 \div 4}{9} = \frac{2}{9}$$

In this problem we divided the numerator by 4.

2. Divide $\frac{3}{7}$ by 5.

Solution: $$\frac{3}{7} \div 5 = \frac{3}{5 \cdot 7} = \frac{3}{35}$$

In this problem, since we cannot divide the numerator by 5 without getting a fraction in the numerator, we multiply the denominator by 5.

3. Divide $\frac{6ax}{a-x}$ by $3a$.

Solution: $$\frac{6ax}{a-x} \div 3a = \frac{6ax \div 3a}{a-x} = \frac{2x}{a-x}$$

In this problem, as in the problems above, we may divide the numerator by $3a$ or multiply the denominator by $3a$. Since $3a$ is a factor of the numerator. the former is preferable.

4. Divide $\frac{3ab}{a-3}$ by $a+2$.

Solution:

$$\frac{3ab}{a-3} \div (a+2) = \frac{3ab}{(a-3)(a+2)} = \frac{3ab}{a^2-a-6}$$

If the divisor and the numerator of the fraction contain a common factor, we may divide the numerator by this common factor and multiply the denominator by the remaining factor of the divisor. For example,

$$\tfrac{12}{7} \div 8 = \tfrac{12}{7} \div (4 \cdot 2) = (\tfrac{12}{7} \div 4) \div 2 = \tfrac{3}{7} \div 2 = \tfrac{3}{14}$$

5. Divide $\frac{x^3-y^3}{x+y}$ by $ax-ay$.

Solution:

$$\frac{x^3-y^3}{x+y} \div (ax-ay) = \frac{(x^3-y^3) \div (x-y)}{(x+y)a} = \frac{x^2+xy+y^2}{ax+ay}$$

Hence, to divide by $ax - ay$, we divide by $x - y$ by dividing the numerator by $x - y$ and divide by a by multiplying the denominator by a.

EXERCISES

Divide, by inspection when possible:

1. $\frac{6}{17}$ by 3

2. $\frac{5}{11}$ by 6

3. $\frac{20}{17}$ by 15

4. $\frac{a^2}{b^3}$ by a

5. $\frac{x^2y}{x}$ by xy

6. $\frac{3a^2x}{5}$ by $2c$

7. $\frac{10a^2b^3}{3c}$ by $2ax$

8. $\frac{24s^3}{k}$ by $6ak$

9. $\frac{a^2-x^2}{a}$ by $a-x$

10. $\frac{5a}{x-2}$ by $x+2$

11. $\frac{4-9y^2}{5}$ by $2+3y$

12. $\frac{a^3-8}{5}$ by a^2-4

13. $\frac{b^2-4b-21}{3m}$ by $b-7$

14. $\frac{4ax+8ab-12ac}{a-7}$ by $4a$

15. $\frac{25x^2-64y^2}{x-1}$ by $5x+8y$

16. $\frac{a^3-x^3}{2my}$ by $a-x$

17. $\frac{a^2-2ax+x^2}{4ac}$ by $a-x$

18. $\frac{3ax}{x^2+4x+16}$ by $ax-4a$

19. $\frac{a^2-5a+6}{3y}$ by a^2-4

20. $\frac{a^3-x^3}{a-x}$ by a^2-x^2

21. $\frac{3a-5}{a-x}$ by $a+2x$

22. $\frac{p^3-64}{a}$ by $ap-4a$

23. $\frac{x^3-125}{3c}$ by $cx^2+5cx+25c$

24. $\frac{4m^2}{16+2m}$ by $16m^2-2m^3$

100. Multiplication of a Fraction by a Fraction.

ILLUSTRATIVE EXERCISES

1. Multiply $\frac{8}{9}$ by $\frac{3}{4}$.

Since a fraction is an indicated division, we are to multiply $\frac{8}{9}$ by 3 and divide by 4.

$$\frac{8}{9} \cdot \frac{3}{4} = \left(\frac{8}{9} \cdot 3\right) \div 4 = \frac{8}{3} \div 4 = \frac{2}{3}$$

We multiplied $\frac{8}{9}$ by 3 by dividing the denominator by 3 and we divided by 4 by dividing the numerator by 4.

The work may be written more compactly as follows:

$$\frac{\overset{2}{\cancel{8}}}{\underset{3}{\cancel{9}}} \cdot \frac{\cancel{3}}{\cancel{4}} = \frac{2}{3}$$

To multiply two or more fractions:

Cancel all the factors common to the numerators and denominators and multiply the remaining factors of the numerators together for a new numerator and the remaining factors of the denominators together for a new denominator.

2. Multiply $\frac{12}{35}$ by $\frac{14}{15}$.

$$\frac{\overset{2}{\cancel{14}}}{\underset{5}{\cancel{15}}} \cdot \frac{\overset{4}{\cancel{12}}}{\underset{5}{\cancel{35}}} = \frac{8}{25}$$

Multiplying by 14 is equivalent to multiplying by 2 and then by 7. $\frac{12}{35}$ is multiplied by 2 by multiplying the numerator by 2 while it is multiplied by 7 by dividing the denominator by 7. Similarly, dividing by 15 is equivalent to dividing by 3 and then by 5. The numerator 12 was divided by 3 and the denominator multiplied by 5.

Had the product been obtained by multiplying the numerators and denominators of the fractions together without cancellation, a correct result would have been obtained, but it would not have been in its simplest form. It is best to use cancellation when possible.

3. Multiply $\frac{a^2x^3}{b^3y^2}$ by $\frac{b^2y^3z}{ax^4}$.

Solution:
$$\frac{\overset{yz}{\cancel{b^2y^3z}}}{\underset{x}{\cancel{ax^4}}}\cdot\frac{\overset{a}{\cancel{a^2x^5}}}{\underset{b}{\cancel{b^3y^2}}}=\frac{ayz}{bx}$$

4. Multiply $\frac{x^2+3x-10}{-2x-24}$ by $\frac{x^2-4x-12}{x^2+2x-15}$.

Solution:
$$\frac{x^2-4x-12}{x^2+2x-15}\cdot\frac{x^2+3x-10}{x^2-2x-24}$$
$$=\frac{\cancel{(x-6)}\,(x+2)}{\cancel{(x+5)}\,(x-3)}\cdot\frac{\cancel{(x+5)}\,(x-2)}{\cancel{(x-6)}\,(x+4)}$$
$$=\frac{(x+2)(x-2)}{(x-3)(x+4)}=\frac{x^2-4}{x^2+x-12}$$

EXERCISES

Multiply, by inspection when possible:

1. $\frac{8}{15}$ by $\frac{5}{6}$

2. $\frac{10}{21}$ by $\frac{14}{25}$

3. $\frac{a^5}{b^3}$ by $\frac{a}{b}$

4. $\frac{3}{x}$ by $\frac{2}{y}$

5. $\frac{8}{x^3}$ by $\frac{x^2}{4}$

6. $\frac{x^3}{y^3}$ by $\frac{y}{bx}$

7. $\frac{9}{a^3b^3}$ by $\frac{2a^3}{3b}$

8. $\frac{a^2x^2}{10}$ by $\frac{x^2}{3a^2}$

9. $\frac{y^3}{z^2}$ by $\frac{z^5}{y}$

10. $\frac{4cy^2}{by}$ by $\frac{9kc}{20y^3}$

11. $\frac{3(x+y)}{5x}$ by $\frac{15x}{8m}$

12. $\frac{m^3p^4}{s^2t^3}$ by $\frac{2st}{3m^2p^4}$

13. $\frac{10t^3}{x^3z^3}$ by $\frac{ax^3z}{5t}$

14. $\frac{a+b}{c}$ by $\frac{x}{y}$

15. $\frac{s+k}{p^2}$ by pq

5

1792

16. $\frac{(m-n)^2}{x^3}$ by $\frac{x^2}{(m-n)^3}$

17. $\frac{x(a+b)}{z}$ by $\frac{z^3}{x^2}$

18. $\frac{x^2-3x}{2a^2}$ by $\frac{5a}{x-3}$

19. $\frac{3m(x-y)}{5x}$ by $\frac{15x}{8m}$

20. $\frac{x(a-b)^2}{z}$ by $\frac{z^3}{x^2(a-b)}$

21. $\frac{a^2-m^2}{10}$ by $\frac{5}{a+m}$ $\frac{a-m}{2}$

22. $\frac{2a}{a^2-4}$ by $\frac{a-2}{6a}$

23. $\frac{m^2-8m+16}{m^2+6m+9}$ by $\frac{m+3}{m^2-4}$

24. $\frac{12n^3a}{5(n^3-a^3)}$ by $\frac{25(n^2+na+a^2)}{10n^2a^2}$

$n-a$

25. $\frac{2(x+3)^2}{5(x+3)(x-7)}$ by $\frac{(x-7)^2}{2(x+3)(x-7)}$

26. $\frac{x^2-9}{x^2-4}$ by $\frac{x-2}{x-3}$

27. $\frac{5x+3}{3x+2}$ by $\frac{9x^2-4}{25x^2-9}$

28. $\frac{x^2-5x+6}{x^2-2x+1}$ by $\frac{x-1}{x-2}$

29. $\frac{x^2-3x-10}{x^2+2x-15}$ by $\frac{x+5}{x-5}$

30. $\frac{x^2-x-6}{x^2+2x-15}$ by $\frac{x+5}{x+2}$

31. $\frac{ax+5x}{3x^2-24x}$ by $\frac{x^2-64}{a^2-2a-35}$

32. $\frac{4s^2}{3x^3}\cdot\frac{-9x}{8s}\cdot\frac{-2x}{s^2}=?$

33. $\frac{3a-1}{a^2-5a+6}$ by $\frac{a^2-11a+18}{9a^2-6a+1}$

34. $\frac{4z^2-9}{15}$ by $\frac{3}{2z-3}$

35. $\frac{-2}{x}\cdot\frac{x^2}{6}\cdot\frac{3y}{x}=?$

36. $\frac{x^2+4x-21}{x^2-x-2}$ by $\frac{x^2+2x-8}{x^2+8x+7}$

37. $\frac{m^2-7m+6}{3m^2+m-2}$ by $\frac{3m^2+7m-6}{m^2+m-6}$

38. $\frac{y^3-8}{y^2-9}$ by $\frac{y^2-y-12}{y^2+2y+4}$

39. $\frac{3ay+6a}{y^3+27}$ by $\frac{2ay+6a}{y^2-4}$

40. $\frac{x^2-2x-63}{x^2-9x-22}$ by $\frac{3x-33}{2x-18}$

41. $\frac{-3}{2-x}\cdot\frac{x^2-4}{12}=?$

42. $\frac{a^2-b^2}{3x}\cdot\frac{9x^2}{a^2-2ab+b^2}\cdot\frac{2}{a+b}=?$

43. $\frac{x+3}{x-2}\cdot\frac{4-x^2}{9-x^2}\cdot\frac{4x-12}{3x+6}=?$

Hint. Change the signs of both terms of the second fraction, writing it $\frac{x^2-4}{x^2-9}$.

44. $\frac{6x^2+x-2}{6x^2+x-35}\cdot\frac{25-4x^2}{1-4x^2}\cdot\frac{6x-14}{9x+6}=?$

45. $\frac{8x^3+27}{9x^2+24x+16}\cdot\frac{3x^2+x-4}{4x^2-6x+9}=?$

46. $\frac{a^3+2a^2+10a+20}{a^2+7a-18}\cdot\frac{a^3+9a^2+2a+18}{a^2+10a}=?$

101. Division of a Fraction by a Fraction.

The method of dividing by a fraction is the same in algebra as in arithmetic. We have had

$$\tfrac{3}{4} \cdot \tfrac{8}{9} = \tfrac{2}{3}$$

Hence, since

$$\text{product} \div \text{multiplier} = \text{multiplicand}$$

$$\tfrac{2}{3} \div \tfrac{3}{4} = \tfrac{8}{9}$$

But $$\tfrac{2}{3} \cdot \tfrac{4}{3} = \tfrac{8}{9}$$

Therefore, by axiom I,

$$\tfrac{2}{3} \div \tfrac{3}{4} = \tfrac{2}{3} \cdot \tfrac{4}{3} = \tfrac{8}{9}$$

Divide $\dfrac{4}{x^2+x-2}$ by $\dfrac{x-1}{x+2}$.

$$\frac{4}{x^2+x-2} \div \frac{x-1}{x+2} = \frac{4}{\underset{x-1}{\cancel{x^2+x-2}}} \cdot \frac{\cancel{x+2}}{x-1} = \frac{4}{(x-1)^2}$$

In this, as in the preceding problem, we found the product of the dividend and a fraction obtained by interchanging the terms of the divisor and multiplying.

Definition. The **reciprocal** of a number is 1 divided by that number.

The reciprocal of 3 is $\frac{1}{3}$. The reciprocal of a is $\frac{1}{a}$. It can readily be shown that the reciprocal of a fraction is the fraction obtained by interchanging its terms. $\frac{3}{2}$ is the reciprocal of $\frac{2}{3}$; the reciprocal of $\dfrac{a}{b}$ is $\dfrac{b}{a}$; the reciprocal of $\dfrac{a-x}{b}$ is $\dfrac{b}{a-x}$, etc.

Dividing a number (integral or fractional) by a fraction is equivalent to multiplying the reciprocal of the fraction by the number.

EXERCISES

Divide, by inspection when possible:

1. 3 by $\frac{2}{5}$ $\frac{15}{2}$
2. $\frac{3}{4}$ by $\frac{5}{6}$ $\frac{5}{8}$
3. $\frac{\overset{3}{\cancel{15}}}{22}$ by $\frac{5}{8}$
4. $\frac{a}{b}$ by $\frac{c}{x}$
5. $\frac{b^2c}{ak}$ by $\frac{bx}{ky}$
6. $\frac{a^2bc}{5}$ by $\frac{3c}{ab}$ $\frac{a^3}{15}$
7. $\frac{a^3s^2}{2y}$ by $\frac{3as^2}{y^2}$ $\frac{a^2}{6}$
8. $\frac{3cyx}{bz}$ by $\frac{2bc}{3y^2x}$
9. $\frac{a^2-9}{6}$ by $\frac{a-3}{3}$
10. $\frac{4}{x^2-4}$ by $\frac{3}{x+2}$
11. 7 by $\frac{3}{5}$ $\frac{35}{3}$
12. 2 by $\frac{2x}{3y}$ $\frac{3y}{x}$
13. am by $\frac{m}{3x}$ $3ax$
14. x^2-25 by $\frac{x-5}{2}$ $2(x+5)$
15. $\frac{5x}{a-x}$ by $\frac{a-x}{8}$ $\frac{40x}{(a-x)^2}$
16. $\frac{m^2n^3}{a^2-x^2}$ by $\frac{m^2n}{a-x}$ $\frac{n^2}{a+x}$
17. $\frac{b^2-4}{b^2-9}$ by $\frac{b+2}{b+3}$
18. $\frac{s^2-4s+3}{s^2-16}$ by $\frac{s-3}{s-4}$
19. $\frac{k^2-k-2}{k+2}$ by $\frac{k-2}{k-1}$
20. $\frac{x^3+y^3}{x^2-2xy+y^2}$ by $\frac{x+y}{x-y}$
21. $\frac{y^2-3y-28}{y^2+2y-35}$ by $\frac{y-7}{y+7}$
22. $\frac{6a^2-a-12}{a^3-8}$ by $\frac{3a+4}{a-2}$
23. $\left(x+\frac{3}{x-4}\right)$ by $\frac{x-3}{x-1}$

    Hint. First combine the terms of the dividend.

24. $\left(2m-\frac{m}{m-2}\right)$ by $\frac{m^2-2m}{k}$
25. $\frac{x+y+z}{y^2z^2}$ by $\left(1+\frac{y^2+z^2-x^2}{2yz}\right)$

$\frac{3}{4}\ \frac{5}{6} =$

26. $\dfrac{27b^3 - 125}{b^2 - 7b + 10}$ by $\dfrac{9ab^2 + 15ab + 25a}{b^2 - 25}$

27. $\dfrac{a^2 - 2ab + b^2 - c^2}{a^2 + 2ab + b^2 - c^2}$ by $\dfrac{ax - bx + cx}{ax + bx - cx}$

28. $\dfrac{a^3 + 3a^2 + a - 5}{2a^2 + a - 10}$ by $\dfrac{a^2 + 4a - 5}{a - 2}$

29. $\dfrac{4x^3 - 12x^2 - x + 3}{x^2 - 16}$ by $\dfrac{2x^2 + 5x - 3}{x^2 - 7x + 12}$

30. $\dfrac{12n^2 + 16n - 3}{4n^2 - 9} \cdot \dfrac{2n^2 + 5n - 12}{30n^2 + n - 1} \div \dfrac{n + 4}{5n + 1} = ?$

31. $\left(1 - \dfrac{1}{x + 1}\right) \div \left(1 + \dfrac{1}{x - 1}\right) = ?$

32. $\left(a - 2 - \dfrac{14}{a + 3}\right) \div \left(a - 1 - \dfrac{21}{a + 3}\right) = ?$

102. Complex Fractions. A complex fraction is one whose numerator or denominator contains a fraction. Any example in the preceding section, if written in fractional form, is a complex fraction. For instance, example 15

$$\frac{5x}{a - x} \div \frac{a - x}{8}, \text{ is } \frac{\dfrac{5x}{a - x}}{\dfrac{a - x}{8}}, \text{ a complex fraction.}$$

A complex fraction requires no further treatment. It may be considered a problem in division. The usual procedure in simplifying a complex fraction is to reduce the numerator and the denominator to their simplest forms and perform the division. Sometimes it is preferable to multiply the numerator and the denominator by the L. C. M. of the denominators.

ILLUSTRATIVE EXERCISE

Simplify $$\frac{\dfrac{2}{x-y}-\dfrac{2}{x+y}}{\dfrac{3}{x^2-y^2}-\dfrac{3}{x+y}}.$$

Solution:

Multiplying the numerator and the denominator by x^2-y^2, the L. C. M. of the denominators, we have

$$\frac{2(x+y)-2(x-y)}{3-3(x-y)}=\frac{2x+2y-2x+2y}{3-3x+3y}$$
$$=\frac{4y}{3-3x+3y}$$

EXERCISES

Reduce the following complex fractions to their simplest forms:

1. $\dfrac{\dfrac{3}{x}}{\dfrac{6}{x^2}}$ $\frac{x}{2}=\frac{x}{2}$ ✓ $\frac{x}{2}$ ✓

2. $\dfrac{5}{\dfrac{10}{a}}$ $\frac{a}{2}$ ✓

3. $\dfrac{x^2}{\dfrac{x}{3}}$, $3x$

4. $\dfrac{\dfrac{a^3}{b}}{a^2}$

5. $\dfrac{1+\dfrac{2}{x}}{1-\dfrac{4}{x^2}}$

6. $\dfrac{a+\dfrac{1}{b}}{a^2-\dfrac{1}{b^2}}$

7. $\dfrac{\dfrac{x^2}{2y}}{3-\dfrac{x}{y}}$

8. $\dfrac{m-\dfrac{n}{2}}{\dfrac{3}{n}}$

9. $\dfrac{b+\dfrac{4}{c}}{b-\dfrac{2}{c}}$

10. $$\frac{1-\dfrac{x^3}{y^3}}{\dfrac{x^2-xy+y^2}{y^2}}$$

11. $$\frac{\dfrac{a}{a+b}+\dfrac{b}{a-b}}{\dfrac{a}{a-b}-\dfrac{b}{a+b}}$$

12. $$\frac{\dfrac{ax^2-x^3}{2a}}{\dfrac{ax-x^2}{6a}}$$

13. $$\frac{\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}}{\dfrac{x^2y^3}{x^2-y^2}}$$

REVIEW EXERCISES

103. 1. How is a fraction defined in algebra?

2. What principle of fractions is made use of in reducing fractions to lower terms?

3. 3 is what part of 4? 1 is what part of 7? $2\frac{1}{2}$ is what part of 4? $1\frac{1}{2}$ is what part of $2\frac{1}{2}$?

4. 1 is what part of a? b is what part of a?

5. m is what part of $x+y$? $x-y$ is what part of $a+b$?

6. $1+m+m^2+\dfrac{m^3}{1+m}=?$

7. $\left(y-3-\dfrac{y^2+9}{y+3}\right)\left(y+3-\dfrac{y^2+9}{y-3}\right)=?$

8. A man earned x dollars in 10 days. What were his daily wages?

9. A man earns \$20 ~~in~~ x days. What are his daily wages?

10. A man runs 100 yards at the rate of x yards a second. What is his time for the 100 yards?

11. A boy runs 220 yards at the rate of $x-3$ yards a second. What is his time?

12. $\left(\frac{12}{x-3} + 1 - \frac{4}{x-1}\right) \cdot \left(\frac{4}{x+1} + 1 - \frac{12}{x+3}\right) = ?$

13. The numerator of a fraction is c and the denominator is 4 greater than the numerator. What is the fraction?

14. What is the cost of n apples at x cents a dozen? At $x + 4$ cents a dozen?

15. A can do a piece of work in 2 days less than B can. If A can do the work in c days, in how many days can B do it? What part of the work can each do in one day?

16. If A can do a piece of work in 5 days and B in 6 days, what part of the work can both do in one day? If A can do it in x days and B in $x + 2$ days, what part of the work can both do in one day?

17. $\frac{m - m^2}{3x^2} \div \frac{2xm - 2xm^2}{7} = ?$

18. $1 + \frac{5}{a - 4} = ?$

19. Simplify the expression $\frac{x}{y^2 + xy} - \frac{y}{x^2 + xy}$.

20. An automobile traveled d miles in t hours. What was the rate per hour? How far did it travel in 3 hours? in k hours?

21. Add $\frac{2}{x-3}$, $\frac{3}{x+3}$, and $\frac{4}{9-x^2}$.

22. From $\left(\frac{x+4}{x-2} - \frac{3x-2}{x-3}\right)$ subtract $\frac{x^2 - 2x - 17}{x^2 - 5x + 6}$.

23. Simplify $$\frac{\dfrac{a}{b^2}+\dfrac{b}{a^2}}{\dfrac{1}{b^2}-\dfrac{1}{ab}+\dfrac{1}{a^2}}.$$

24. Explain what is meant by the three signs of a fraction.

25. Solve the following equations by factoring:

(a) $x^2 - 2x - 15 = 0$
(b) $x^2 + 8x - 65 = 0$
(c) $2x^2 + 11x - 40 = 0$
(d) $5k^2 - 28k - 96 = 0$

SUMMARY

SECTION
92. Definition of a fraction as an indicated division.
93. Multiplication and division of the terms of a fraction by the same number.
94. The three signs of a fraction.
95. Reduction of fractions to lowest terms.
96. Reduction of fractions to a common denominator.
97. Addition and subtraction of fractions.
98. Multiplication of a fraction by an integral expression.
99. Division of a fraction by an integral expression.
100. Multiplication by a fraction.
101. Division by a fraction.
102. Complex fractions.
103. Review exercises.

CHAPTER XVI

EQUATIONS CONTAINING FRACTIONS

104. If an equation contains fractions, the first step in the solution is usually to clear of fractions by multiplying both members of the equation by the L. C. M. of the denominators.

ILLUSTRATIVE EXERCISE

Solve the equation,

$$\frac{x}{4}+\frac{2x-3}{5}-\frac{3x+2}{10}=\frac{3}{5}$$

Solution:

Multiplying both members of the equation by 20, the L. C. M. of the denominators,

$$5x + 4(2x - 3) - 2(3x + 2) = 12$$

Removing parentheses, $5x + 8x - 12 - 6x - 4 = 12$

Collecting terms, $7x - 16 = 12$

Adding 16 to each member, $7x = 28$

Dividing each member by 7, $x = 4$

Check: $\frac{4}{4}+\frac{8-3}{5}-\frac{12+2}{10}=1+1-\frac{7}{5}=\frac{3}{5}$

In handling a fraction, such as $-\frac{3x+2}{10}$ above, care should be taken to have the signs correct and, for a time at least, it will be well when clearing of fractions to write the numerators in parentheses as in the first step above.

EXERCISES

Solve the following equations, by inspection when possible. Check when the root is an integer.

1. $\frac{x}{2} = 3$

2. $\frac{3x}{5} = 3$

3. $\frac{m}{2} + \frac{m}{3} = 5$ m = 6

4. $\frac{r}{3} - \frac{5r}{6} = -3$

5. $\frac{5s}{2} - \frac{3s}{4} = 7$

6. $\frac{s}{2} - \frac{s}{3} = \frac{s}{4} - 1$

7. $\frac{10}{y} = 2$

8. $\frac{27}{b} = 9$

9. $\frac{12}{x} - \frac{9}{x} = 1$

10. $\frac{x}{2} + \frac{3x+2}{5} - \frac{2x}{3} = \frac{5}{6}$

11. $\frac{3x+5}{4} - \frac{2x+3}{2} = -\frac{1}{2}$

12. $\frac{5x-1}{3} + \frac{2x+1}{5} = \frac{9}{10}$

13. $\frac{2x+4}{5} - \frac{7x-6}{15} = 2$

14. $\frac{x}{a} = 1$

15. $\frac{x+3}{4} = 2$

16. $\frac{m-2}{3} = \frac{m}{4}$

17. $\frac{6}{k} = \frac{8}{k+1}$

18. $\frac{12}{3h} + 3 = \frac{4}{2h} + 4$

19. $\frac{3}{2x} - \frac{1}{3x} = \frac{7}{3}$

20. $\frac{3}{x+1} = \frac{2}{x+2}$

21. $\frac{2x+3}{5} = \frac{4x-1}{3}$

22. $\frac{x+1}{8} + \frac{x+2}{3} = \frac{2x+2}{4}$

23. $\frac{3}{x} + \frac{5}{2x} = \frac{5}{x+2}$

24. $\frac{4a-3}{2a-1} = \frac{4a-7}{2a-5}$

25. $\frac{m-2}{m-7} + \frac{m-4}{m-6} = 2$

26. $\frac{4}{x-1} + \frac{6}{x+1} = \frac{3}{x^2-1}$

27. $c - \frac{2c + 7}{3} = \frac{5c + 1}{2} - 5$

28. $\frac{3x + 4}{2} - \frac{x}{3} + \frac{5x - 3}{6} = 3$

29. $\frac{x - 7}{2} - 2 + \frac{3x + 7}{4} = 20$

30. $\frac{x}{5} + \frac{2x}{3} + \frac{2 - x}{3} = \frac{3 + x}{15}$

31. $\frac{2x - 5}{6} + \frac{x}{3} - \frac{x - 4}{4} = 2$

32. $\frac{5}{3 - x} + \frac{2}{3 + x} = \frac{10}{9 - x^2}$

33. $\frac{3x - 4}{x^3 - 1} + \frac{x + 3}{x^2 + x + 1} = \frac{1}{x - 1}$

34. $\frac{3}{2 - x} - \frac{2}{2 + x} = \frac{3x}{4 - x^2}$

Sometimes it is preferable to combine two or more of the fractions before clearing of fractions.

ILLUSTRATIVE EXERCISE

Solve: $\frac{x - 2}{2} - \frac{3x - 7}{6} = \frac{x + 1}{2x - 3}$

Solution:

Combining the fractions in the first member of the equation, $\frac{1}{6} = \frac{x + 1}{2x - 3}$

Clearing of fractions, $2x - 3 = 6x + 6$

Subtracting $6x - 3$ from each member, $-4x = 9$

whence, $x = -\frac{9}{4}$

Check: $$\frac{-\frac{9}{4} - 2}{2} - \frac{3\left(-\frac{9}{4}\right) - 7}{6} = \frac{-\frac{9}{4} + 1}{2\left(-\frac{9}{4}\right) - 3}$$

or, $-\frac{17}{8} - \left(-\frac{55}{24}\right) = \frac{-\frac{5}{4}}{-\frac{30}{4}} = \frac{5}{30} = \frac{1}{6}$ whence, $\frac{1}{6} = \frac{1}{6}$

Began Her

EXERCISES

Solve the following equations:

1. $\dfrac{2x+1}{3} - \dfrac{8x-3}{12} = \dfrac{x-2}{2x+1}$ (4y+12)

2. $\dfrac{5y+4}{10} - \dfrac{y+1}{2} = \dfrac{2-y}{3-y}$

3. $\dfrac{3y-5}{6} - \dfrac{y+3}{2} = \dfrac{1}{y+3}$ 24x+12

4. $\dfrac{6k+7}{3} - \dfrac{4k+1}{2} = \dfrac{5}{k+1}$ 6(k+1)

5. $\dfrac{7x-4}{10} - \dfrac{2x-3}{5} = \dfrac{3x^2+2x+9}{10(x+3)}$

6. $\dfrac{x-1}{x-2} - \dfrac{x-3}{x-4} = \dfrac{x-2}{x-3} - \dfrac{x-4}{x-5}$

7. $\dfrac{4}{1+b} + \dfrac{b+1}{1-b} - \dfrac{b^2-3}{1-b^2} = 0$

8. $2 - \dfrac{7s-1}{6} = 3s - \dfrac{19s+3}{4}$

9. $\dfrac{2}{y-1} - \dfrac{5}{2(y-1)} = \dfrac{8}{3(y-1)} - \dfrac{y}{y-1} + \dfrac{5}{18}$

10. $\dfrac{1}{z-1} - \dfrac{1}{z-2} = \dfrac{1}{z-3} - \dfrac{1}{z-4}$

11. $6m - \dfrac{27m-1}{4} - \dfrac{2(4m-1)}{5} = \dfrac{9m-5}{4} - \dfrac{11m-2}{3} + 22$

105. Literal Coefficients.

1. Solve the equation $ax + b = c$

Solution:

Subtracting b from each member, $ax = c - b$

Dividing by a, $x = \dfrac{c-b}{a}$

It will be noticed that the method of solving this equation is the same as that in section 38.

2. Solve the equation $\frac{x}{a} + \frac{3x}{b} - 2 = \frac{cx}{2}$.

Solution:

Multiplying by $2ab$, the L. C. M. of the denominators,

$$2bx + 6ax - 4ab = abcx$$

Subtracting $abcx - 4ab$ from each member,

$$2bx + 6ax - abcx = 4ab$$

Factoring the first member,

$$(2b + 6a - abc)x = 4ab$$

Dividing by $2b + 6a - abc$,

$$x = \frac{4ab}{2b + 6a - abc}$$

EXERCISES

Solve the following equations for x, mentally when possible.

1. $cx = m$ x
2. $ax - 2 = k$ $x = \frac{k+2}{a}$
3. $2x - ax = v$ $x = \frac{v}{2-a}$
4. $kx = ax + b$ $x = \frac{b}{k-a}$
5. $cx + 2x = a - b$
6. $\frac{3x}{c} + \frac{x}{b} = 2$
7. $5x + cx = 10$
8. $ax + cx = b$
9. $ax - kx = 6$
10. $\frac{ax}{2} + \frac{bx}{3} = m$
11. $5x - cx = a$
12. $(3a + 2b)x = s$
13. $ax - 2ab + bx = a^2 + b^2$
14. $\frac{x}{2} + \frac{x}{a} = 3$
15. $cx - 5 = ax + 3$
16. $m + ax = n - 2x$
17. $\frac{1}{2x} + \frac{1}{x} = \frac{b}{x^2}$
18. $\frac{x-3}{a} + \frac{x}{b} = \frac{x+2}{3}$
19. $\frac{2}{a+x} + \frac{3}{a-x} = \frac{2a}{a^2 - x^2}$

Solve each of the following equations for each letter:

20. $i = prt$

21. $A = bh$

22. $C = 2\pi r$

23. $A = \dfrac{h(b_1 + b_2)}{2}$

Suggestion: b_1 and b_2 are different numbers.

24. $d = rt$

25. $s = 2\pi rh$

26. $a = gt$

27. $C = \dfrac{5(F - 32)}{9}$

106. Problems Leading to Fractional Equations.

1. The sum of two numbers is 31. If the greater be divided by the smaller, the quotient is 4 and the remainder 1. Find the numbers.

Solution:

Let $x =$ the smaller number.
Then $31 - x =$ the larger number.

By the conditions of the problem,

$$\frac{31 - x}{x} = 4 + \frac{1}{x}$$

Multiplying by x, $31 - x = 4x + 1$
Solving, we find $x = 6$
whence, $31 - 6 = 25$

Hence, the two numbers are 6 and 25. Since 25 divided by 6 gives a quotient 4, and a remainder 1, these results are correct.

2. The sum of two numbers is 54 and their quotient is 8. What are the numbers?

3. The difference between two numbers is 87 and the greater divided by the smaller gives a quotient of 7 and a remainder of 9. Find the numbers.

4. What number must be added to the numerator and the denominator of the fraction $\frac{5}{7}$ so that the resulting fraction shall be equal to $\frac{5}{6}$?

5. If 5 be subtracted from both terms of a fraction whose numerator is 2 less than its denominator, the resulting fraction will be equal to $\frac{5}{6}$. What is the fraction?

6. The tens' digit of a number exceeds the ones' digit by 1. If the number is divided by the sum of its digits, the quotient is 6. What is the number?

7. In a number of two digits the tens' digit is one greater than the ones' digit. The fraction the numerator of which is the given number and the denominator is the number obtained by reversing the order of the digits is equal to $\frac{7}{4}$. What is the number?

8. A can do a piece of work in 2 days less time than B can. Together they can do the work in $3\frac{3}{7}$ days. In how many days can each do the work alone?

Solution:

Let $x =$ the number of days it takes A alone to do the work.

Then, $x + 2 =$ the number of days it takes B alone to do the work.

$\frac{1}{x} =$ the part of the work A can do in one day.

$\frac{1}{x+2} =$ the part of the work B can do in one day.

Then, $$\frac{1}{x} + \frac{1}{x+2} = \frac{1}{3\frac{3}{7}} = \frac{7}{24}$$

Multiplying both members by $24x(x+2)$, $$24(x+2) + 24x = 7x(x+2)$$

Removing parentheses and collecting terms, $7x^2 - 34x - 48 = 0$

Factoring the first member, $(7x + 8)(x - 6) = 0$

Whence, $7x + 8 = 0$

and $x = -\frac{8}{7}$

or, $x - 6 = 0$

and $x = 6$

The solution $-\frac{8}{7}$ is a solution of the equation, but since it is inconsistent with the statement of the problem, it is not the desired solution.

If $x = 6$, then $x + 2 = 8$. Hence A can do the work in 6 days and B in 8 days. Since $\frac{1}{6} + \frac{1}{8} = \frac{7}{24}$ and

$$\frac{24}{24} \div \frac{7}{24} = \frac{24}{7} = 3\frac{3}{7}$$

the conditions of the problem are satisfied and $x = 6$ is the desired solution.

9. A can do a piece of work in 4 days less time than B can and both can do the work in $4\frac{4}{5}$ days. In how many days can each do the work alone?

10. A can do a piece of work in 5 days less time than B can. After B has worked 3 days alone, A can finish the work in 8 days. How long will it take each alone to do the work?

Suggestion. If x represents the number of days required for A to do the work, the conditions of the problem give the equation,

$$\frac{8}{x} + \frac{3}{x+5} = 1$$

What does the $\frac{8}{x}$ represent? the $\frac{3}{x+5}$? the 1?

11. A can do a piece of work in 2 days less time than B. Both work at the job for 5 days and then B finishes it in 1 day. In how many days can each do the work alone?

12. By the last census the population of New Haven, Connecticut, exceeds that of Duluth, Minnesota, by approximately 60,000. If the population of New Haven is divided by that of Duluth diminished by 10,000, the quotient is 2. What is the population of each city?

13. The interest on a certain sum for one year at a certain rate is $25. If the rate of interest were one per cent greater, the interest would be $5 greater. What are the rate and the principal?

14. An automobilist finds that on a certain road a gallon of gasoline carries him 4 miles less than the gasoline costs cents per gallon. The cost of gasoline per mile is $1\frac{1}{4}$ cents. What is the cost of gasoline per gallon?

15. A man walked 24 miles and rode back, the entire trip occupying 11 hours. His rate of walking was 5 miles per hour less than his rate of riding. How fast did he walk?

16. A railway and a highway extend side by side. A passenger train leaves a certain station two hours later than an automobile going in the same direction. The train runs 10 miles an hour faster than the automobile and overtakes the automobile when 75 miles from the station. At what rate does each travel?

17. An automobilist and a bicyclist leave A for B at the same time. The automobilist reaches B and returns immediately, meeting the bicyclist 25 miles from B. If A and B are 100 miles apart and the automobilist travels 10 miles an hour faster than the bicyclist, what is the rate of each?

18. If in problem 17 the automobilist travels $4\frac{1}{2}$ miles an hour faster than the bicyclist and leaves A $4\frac{1}{2}$ hours after the bicyclist does, they will arrive at B at the same time. What is the rate of each?

19. A and B start 80 miles apart and travel toward each other. A's rate is 1 mile an hour faster than B's. A is delayed 2 hours and when they meet they have both traveled the same distance. What is the rate of each?

20. A farmer has 20 acres more of oats than of corn. The yield of corn is double that of the oats and the total yield of each crop is 1200 bushels. What is the yield of each per acre?

REVIEW EXERCISES

107. **1.** State the rule for adding or subtracting fractions having the same denominator. 8x − 1

2. Combine into one fraction $3-\frac{x+2}{2x}+\frac{x-1}{2x}$. 7

3. State the rule for adding or subtracting fractions having different denominators.

4. Combine $\frac{2a-3}{3a}-(a+3)+\frac{a-5}{a+2}$.

5. How is a fraction multiplied by an integral expression?

6. Multiply $\frac{3m-2}{m^2-25}$ by $am+5a$. 5) $\frac{6}{a}$

7. Multiply $\frac{3x}{2b}$ by $2b^2-4b$. 5x

8. If b apples cost a cents, what is the cost of one apple? 5 apples?

9. If eggs are worth x cents a dozen, what is the cost of 1 egg? 8 eggs? k eggs?

10. How is a fraction divided by an integral expression?

11. Divide (a) $\frac{a}{c}$ by $2x$.

(b) $\frac{x^2-12x+20}{x-7}$ by $x-2$.

12. What is the cost of k oranges at m cents a dozen and p bananas at p cents a dozen?

13. State the rule for multiplying by a fraction.

14. Multiply $\frac{2ax^2}{4m^2}$ by $\frac{3x^3m}{10a^2c}$.

$\frac{3}{20cm}$

5a

15. $\frac{a+3}{x} \cdot \frac{3x^2 - 1}{a^2 - 9} \cdot \frac{ax - 3x}{9a} = ?$

$\frac{x+3}{9a}$

16. State the rule for dividing by a fraction.

17. Divide $\frac{6a^2x}{11b^2y}$ by $\frac{3ax}{2b}$.

$\frac{4a}{11y}$

18. Divide $\frac{x^3 - 27y^3}{4x^2 - 25}$ by $\frac{x^2 + 3xy + 9y^2}{4x - 10}$.

19. Solve: $\frac{8}{x-2} - \frac{14}{x+3} = \frac{28}{x^2 + x - 6}$.

20. The numerator of a certain fraction is 2 greater than the denominator. The fraction exceeds its reciprocal by $\frac{5}{6}$. What is the fraction?

21. $\left(\frac{a+b}{a-b} - \frac{a-b}{a+b}\right) \div \left(\frac{6}{a} + \frac{3}{a-b} - \frac{3}{a+b}\right) = ?$

22. Simplify $\dfrac{\dfrac{1}{m-n} - \dfrac{m}{m^2 - n^2}}{\dfrac{m}{mn + n^2} - \dfrac{n}{m^2 + mn}}$.

SUMMARY

SECTION

104. Fractional equations.
105. Literal coefficients.
106. Problems leading to fractional equations.
107. Review exercises.

ruler

CHAPTER XVII

GRAPHS

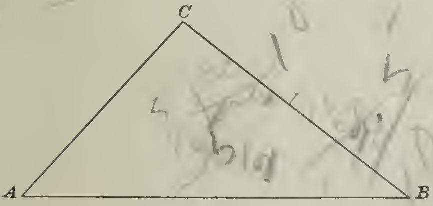
108. Graphical Representation of Distances.

The student is familiar with the use of lines to represent distances. In section 26 a certain unit of length was used to represent a mile. This is constantly done by engineers, architects, and others who are dealing with distances that it would be impossible or inconvenient to draw on paper.

If a line 1 inch long represents a mile, what will a line 5 inches long represent? one $7\frac{1}{2}$ inches long?

If a line 1 inch long represents 100 rods, what will a line 3 inches long represent? one $4\frac{5}{16}$ inches long?

If 1 inch represents 10 feet, draw lines representing 5 feet, 20 feet, 35 feet, $12\frac{1}{2}$ feet, $17\frac{1}{2}$ feet.



If 1 inch represents 100 rods, what are the lengths of the sides of the triangle ABC?

In figure 1 on the next page, the length of one of the small spaces represents 100 miles and the lines represent the distances of the cities from New York. What is the distance of each city from New York?

This representation of the distances is called a **graph** and by means of it relative distances are more easily grasped by the mind.

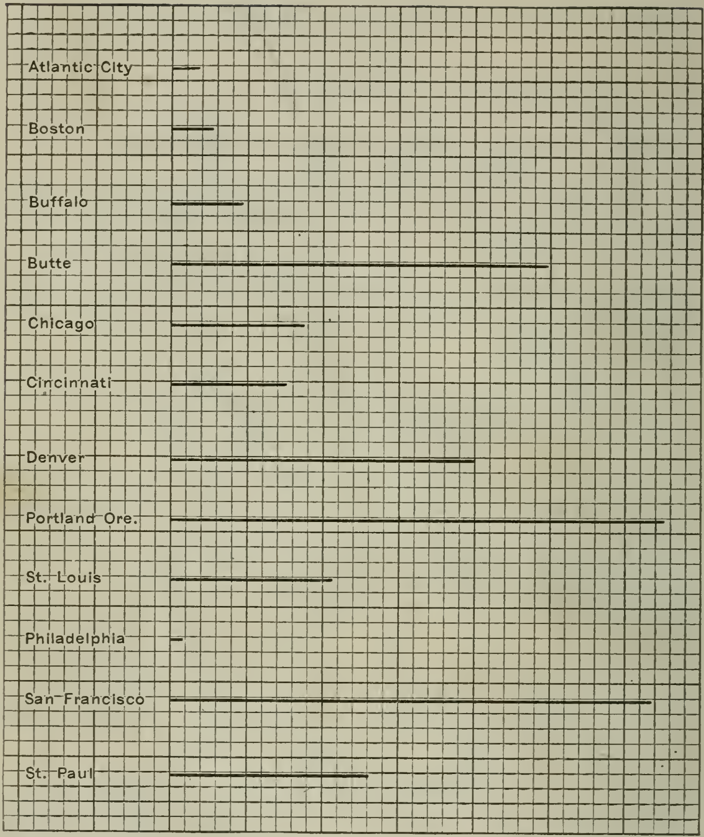


Figure 1. Graph showing relative distances of various cities of the United States from New York. The length of small space represents 100 miles.

109. Graphical Representation of Other Number Relations.

The graph is not confined to representing relations between distances, but may be used to picture other

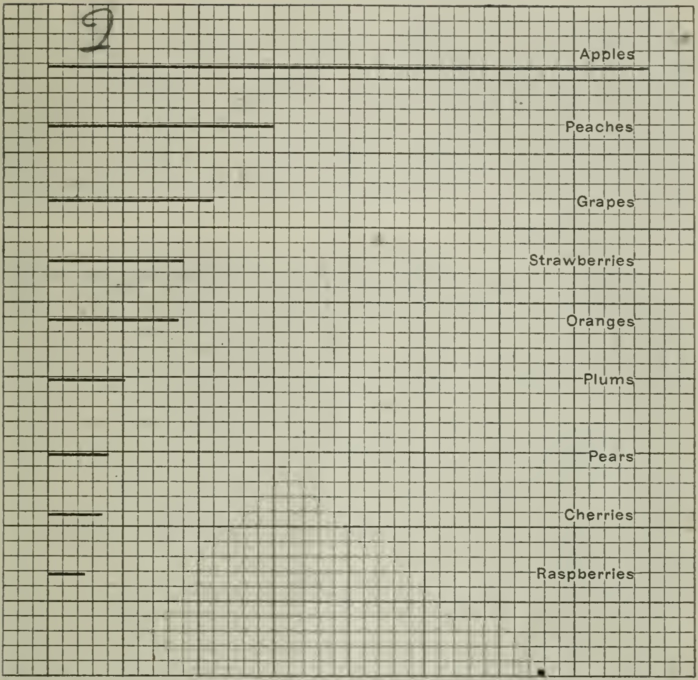


Figure 2. Graph showing the relative values of the various fruit crops grown in the United States in 1909. The length of a small square represents 2 million dollars.

number relations. According to the last census (1910), the value of the fruit crops grown in the United States in 1909 was as follows (see figure 2):

40

9

Apples,	80 million dollars
Peaches,	30 million dollars
Grapes,	22 million dollars
Strawberries,	18 million dollars
Oranges,	17½ million dollars
Plums,	10 million dollars
Pears,	8 million dollars
Cherries,	7 million dollars
Raspberries,	5 million dollars

In figure 2, two million dollars is represented by a line the length of one unit space. The value of the apples is therefore represented by a line 40 times as long, the value of the peaches by a line 15 times as long, the value of the grapes by a line 11 times as long, etc. The relative values of the crops listed below the figure will then be represented by the graph shown.

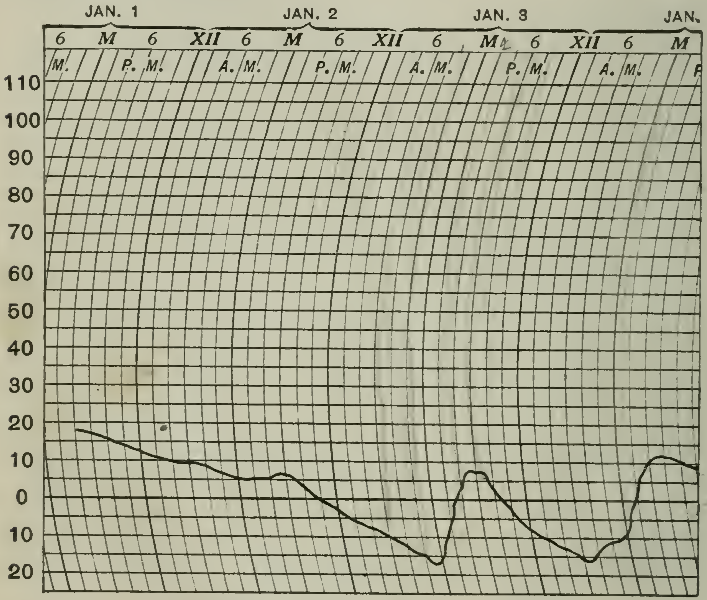


Figure 3

110. Temperature and other Graphs.

The graph is used, also, to show how one quantity changes with another, such as temperature with the time of the day, population with the year, etc. In figure 3, we have a copy of the graph of the temperature at Dubuque, Iowa, made by the recording thermometer, or thermograph.

To make a graph of this kind, two base lines, or **axes**, are needed. A horizontal line is chosen as one axis and a vertical line as the other. If all the numbers involved are positive, the axes are usually taken at the bottom and the left side of the figure. If positive and negative numbers are used, the axes are taken through the middle of the figure.

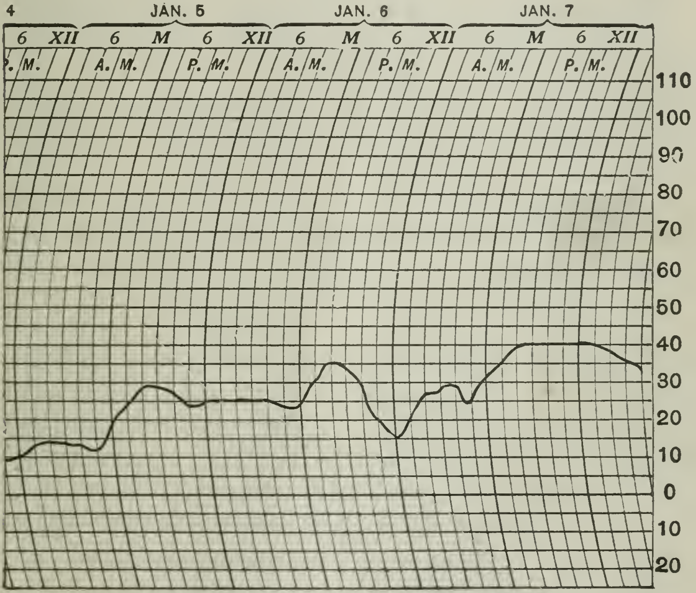


Figure 3—continued

In the graph given above, the hours are marked along a horizontal line (at the top) and the degrees along a curved line nearly vertical. Since the hours are positive, the vertical axis is at the left of the figure. The degrees are positive and negative. Hence, the horizontal axis is taken through the middle of the figure so as to leave spaces above and below.

Read from the graph (figure 3) the temperature on January 3, at

Noon. 2:00 P.M. 3:00 P.M. 5:00 P.M. 6:00 P.M.
9:00 A.M. 5:00 P.M. 10:00 A.M. 8:00 P.M. 2:00 A.M.

At what time of the day on January 4 was the temperature 0° 10° – 15° 11°
– 10° 5° – 5° 4°

Represent graphically the data in the following problems:

1. The altitudes of the principal mountains in the United States are as follows:

Mt. McKinley	20,300 ft.	Mt. Washington,	6,300 ft.
Mt. Whitney,	14,500 ft.	Mt. Rainier,	14,500 ft.
Mt. Elbert,	14,400 ft.	Mt. Hood,	12,500 ft.
Mt. Katahdin,	5,200 ft.	Kings Peak,	13,500 ft.

Let the length of a small square represent 1000 feet.

2. The lengths of some of the principal rivers of the United States are approximately as follows:

Arkansas,	2,150 miles	Mississippi,	3,160 miles
Colorado,	1,350 miles	Missouri,	3,100 miles
Columbia	750 miles	Ohio,	950 miles
Hudson,	280 miles	Red,	1,600 miles
James,	450 miles	Yukon,	2,000 miles

Let the length of a small square represent 100 miles.

3. The production of steel in the principal countries of the world for the year 1910 was approximately as follows:

Austria-Hungary,	2,150,000 tons
Belgium,	1,450,000 tons
France,	3,500,000 tons
Germany,	15,700,000 tons
United States,	26,500,000 tons
Russia,	2,350,000 tons
United Kingdom,	6,100,000 tons
Italy,	625,000 tons

Let the length of a small square represent 500,000 tons.

Draw a temperature graph from the following data, which are from the report of the Weather Bureau at Chicago, Dec. 12, 1912:

1:00 A.M.	9°	9:00 A.M.	8°	5:00 P.M.	20°
2:00 A.M.	6°	10:00 A.M.	9°	6:00 P.M.	19°
3:00 A.M.	6°	11:00 A.M.	11°	7:00 P.M.	20°
4:00 A.M.	4°	Noon	13°	8:00 P.M.	20°
5:00 A.M.	3°	1:00 P.M.	17°	9:00 P.M.	21°
6:00 A.M.	4°	2:00 P.M.	18°	10:00 P.M.	22°
7:00 A.M.	5°	3:00 P.M.	20°	11:00 P.M.	22°
8:00 A.M.	6°	4:00 P.M.	20°	Midnight	21°

Mark the hours on a line near the bottom or top of the page and the degrees of temperature from 20° or more below zero to 30° or more above zero on a vertical line at the left of the page.

On the vertical line passing through 1 A.M. make a mark at the proper place to indicate 9° below zero. On the vertical line through 2 A.M. make a mark at the proper place to indicate 6° above zero, and continue until a mark has been made for each temperature given. Then draw a smooth curve connecting these points.

Construct a temperature graph from the following data:

Midnight	− 1°	8:00 A.M.	+ 3°	4:00 P.M.	+ 12°
1:00 A.M.	− 3°	9:00 A.M.	+ 4°	5:00 P.M.	+ 9°
2:00 A.M.	− 5°	10:00 A.M.	+ 5°	6:00 P.M.	+ 7°
3:00 A.M.	− 5°	11:00 A.M.	+ 7°	7:00 P.M.	+ 5°
4:00 A.M.	− 4°	Noon	+ 9°	8:00 P.M.	+ 4°
5:00 A.M.	− 4°	1:00 P.M.	+ 11°	9:00 P.M.	+ 2°
6:00 A.M.	− 2°	2:00 P.M.	+ 13°	10:00 P.M.	+ 1°
7:00 A.M.	+ 1°	3:00 P.M.	+ 14°	11:00 P.M.	0°

Construct a population graph for the United States from the following data:

Year	Population	Year	Population	Year	Population
1800	5,300,000	1840	17,000,000	1880	50,000,000
1810	7,250,000	1850	23,200,000	1890	62,000,000
1820	9,600,000	1860	31,400,000	1900	76,000,000
1830	12,900,000	1870	38,500,000	1910	92,000,000

As all the data in this problem are positive, the axes may be taken at the bottom and left side of the page. Mark the years from 1800 to 1910 on the horizontal axis. On the vertical axis let the length of the side of a small square represent one million of population.

The population in 1810 will be indicated by a mark on the vertical line passing through 1810, $7\frac{1}{4}$ spaces above the horizontal axis. Make a mark for each of the given populations and connect them.

Tell from your graph the approximate population in 1825; in 1845; in 1883.

When was the population 25 million? 84 million?

Construct from the following data a graph showing the growth in the number of savings banks in the United States:

Year	Banks	Year	Banks
1840	60	1880	625
1850	110	1890	920
1860	275	1900	1000
1870	520	1910	1760

111. Graph of an Equation.

The relative values of x and y in such an equation as

$$y - 2x = 1$$

may be pictured by means of a graph.

To construct the graph of this equation, two axes are chosen, one horizontal and the other vertical, as in the graph showing temperature. The intersection of these axes is called the origin, and is usually denoted by the letter O. The values of x are measured on the horizontal axis, measuring to the right or left from the origin, hence this axis is called the x-axis and is lettered XX' in the figure. The values of y are measured on the vertical axis, measuring up or down from the origin, hence this axis is called the y-axis and is lettered YY' in the figure. Positive values of x are measured to the right from the origin and negative values to the left. Positive values of y are measured upward from the origin and negative values downward.

Substituting any value for x in the above equation (let us say, 1), we have

$$\text{When } x = 1,\ y = 3$$

First, the point indicating the value of x is fixed on the x-axis; this is one unit space to the right of the origin. Then the point indicating the value of y is fixed on the y-axis; this is 3 unit spaces above the origin. The point of intersection of lines running through these two points and parallel to the two axes gives us one point (the point marked A) on our graph.

(A quicker way to fix point A is to measure 1 unit space to the right of point O and then 3 unit spaces upwards, but the reason for it will not be understood until the preceding paragraph is studied.)

To get another point on our graph, substitute some other value for x (let us say, 2) in the same equation. Then if $x = 2$, $y = 5$. The point indicating these values is found by measuring 2 unit spaces to the right on the x-axis and upwards 5 unit spaces on a line parallel to the y-axis, giving the point marked B in the figure.

If $x = -1$, $y = -1$. The corresponding point on the graph is found by measuring 1 unit space to the left of the

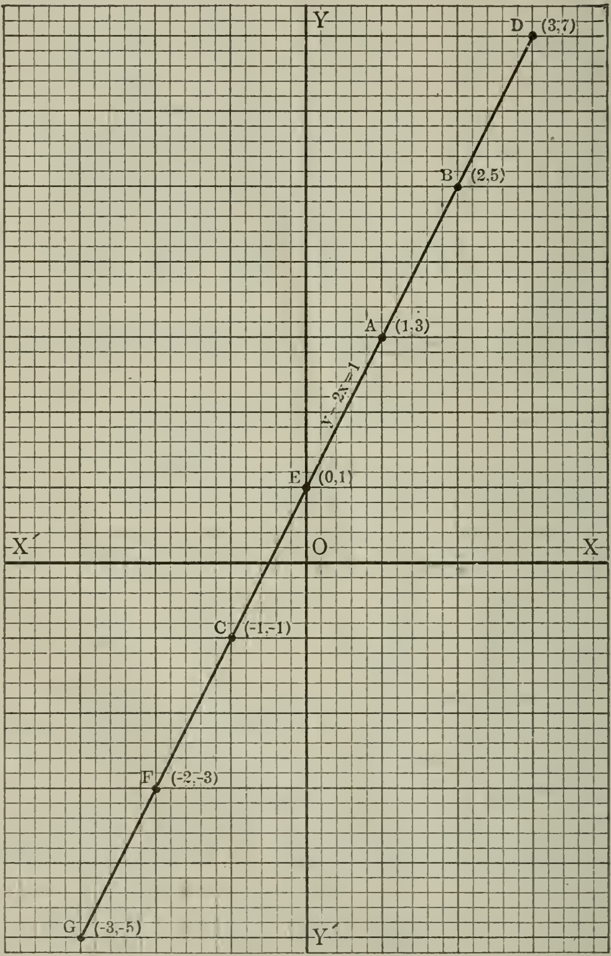


Figure 4. Graph of the equation $y - 2x = 1$.

origin on the x-axis and then 1 unit space downwards, giving the point marked C in the figure.

If $x = -2$, what is y? Find the corresponding point.

If $x = 0$, what is y? Find the corresponding point.

If $x = -\frac{1}{2}$, what is y? Find the corresponding point.

These values of x and y may be conveniently arranged in columns as follows:

x	y
0	1
1	3
2	5
3	7
4	9
-1	-1
-2	-3
-3	-5

That is, when $x = 0$, $y = 1$; when $x = 1$, $y = 3$; etc.

What appears to be the relation between the points corresponding to these pairs of values of x and y?

If we draw a line connecting these points (see figure 4), we have the graph of the equation $y - 2x = 1$. What do you notice about this line?

The point A in figure 4 is frequently referred to as "the point (1, 3)," the first number denoting the x value and the second the y value. B is the point (2, 5); C is the point $(-1, -1)$, etc.

Draw the graph of the equation $x + 2y = 6$.

As in the preceding problem, we find the following pairs of values of x and y. We find the first point (0, 3) by measuring 3 units upwards from the origin on the y-axis; the second point (2, 2) by measuring 2 units to the right of the origin on the x-axis and then 2 units upwards; the fourth point $(-2, 4)$, by measuring 2 units to the left of the origin on the x-axis and then 4 units upwards; etc. Connecting these points by a line, we have the graph of the equation $x + 2y = 6$.

x	y
0	3
2	2
4	1
-2	4
-4	5

3'20

The graphs in the two cases given appear to be straight lines, and by elementary geometry it is easily proved that the graph of any equation of the first degree in two unknowns is a straight line. For this reason an equation of the first degree is called a **linear** equation.

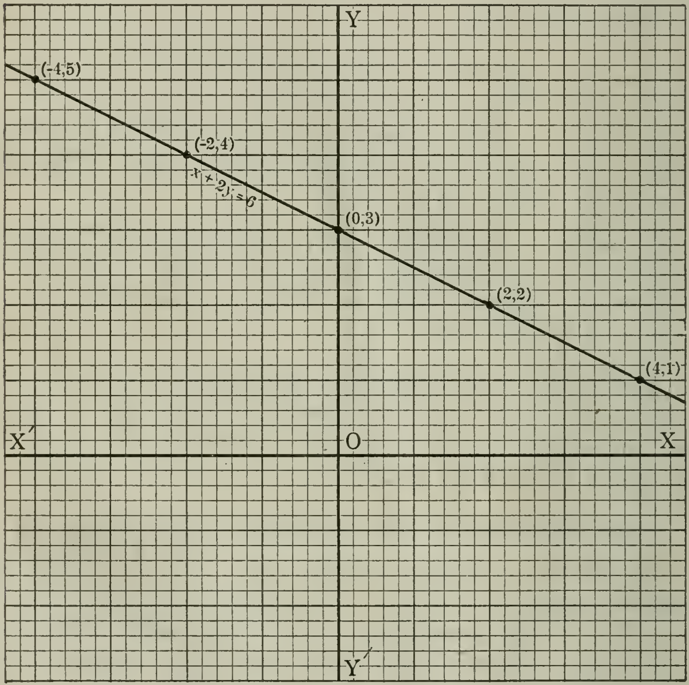


Figure 5. Graph of the equation $x + 2y = 6$.

Since a straight line can be drawn if two of its points are known, the graph of an equation of the first degree in x and y can be drawn by finding two points on the graph and drawing the straight line connecting them.

Draw the graph of $2x - 3y = 6$.

If $x = 0$, $y = -2$; if $x = 3$, $y = 0$. Locate the point (0, 2) and the point (3, 0), and draw the straight line connecting them. The line *AB*, figure 6, is the graph of $2x - 3y = 6$.

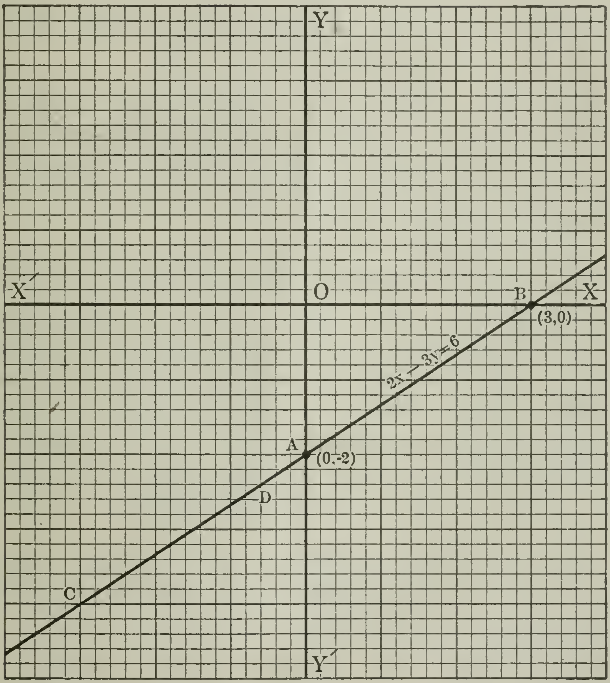


Figure 6. Graph of the equation $2x - 3y = 6$.

From the graph in figure 6 it is possible to determine other solutions of the equation $2x - 3y = 6$ than the two used in drawing the graph. Select any point on the graph and find the x value and the y value corresponding to that point. These values are a solution of the equation. For example, C is the point $(-3, -4)$. Hence $x = -3$, $y = -4$ is a solution of the given equation. Substituting these values in the equation, we have

$$2(-3) - 3(-4) = -6 - (-12) = 6$$

Also, D is the point $(-\frac{3}{5}, -\frac{12}{5})$. Hence $x = -\frac{3}{5}$, $y = -\frac{12}{5}$ is a solution of $2x - 3y = 6$.

In finding solutions by measurement, however, we may be unable to draw our figures precisely enough to determine exactly the x and y values corresponding to a given point. In that case, the solution found may be only approximately correct.

It is evident that there is an indefinitely large number of solutions, every point on the line giving a solution. We saw this in the first two equations whose graphs we drew. In each case we found several sets of values of x and y that satisfied the equation. And we may find as many more as we please, since for every value we may give x we can find a value of y that will satisfy the equation. Hence x and y are called **variables.**

Draw the graph of each of the following equations and obtain from each graph three solutions of the equation:

1. $2y + x = 3$
2. $y = x + 5$
3. $y = x + 1$
4. $y = x - 2$
5. $x = 6 - 2y$
6. $x = 3y - 5$
7. $3x - y = 5$
8. $y - 4x = 1$
9. $2x + y = 4$
10. $5x + 2y = 10$
11. $3x - 2y = 7$
12. $3y + x = 2$
13. $4x - 5y = 20$
14. $7y + 4x = 14$
15. $4y + 6x = 12$

REVIEW EXERCISES

112. 1. Plot the points (0, 0), (0, 1), (0, 2), (0, 4), (0, − 1), and (0, − 5). What do you notice about all these points?

2. Plot the graph of the equation $2x + 3y = 10$ and find from the graph four solutions of the equation.

3. Plot the points (2, 0), (2, 2), (2, 3), (2, $\frac{1}{2}$), (2, − 3), and (2, − 6). What relation exists among these points?

4. Plot the points, (− 2, 3), (4, 3), (4, − 2), and (− 2, − 2). If the points are connected by straight lines in the order given, what kind of figure is formed?

5. Represent graphically the following data, which give the approximate number of miles of railroad in operation in the United States during the years 1900 to 1910 inclusive:

1900	192,000 miles	1906	218,000 miles
1901	196,000 miles	1907	225,000 miles
1902	197,000 miles	1908	228,000 miles
1903	207,000 miles	1909	235,000 miles
1904	211,000 miles	1910	240,000 miles
1905	213,000 miles		

SUMMARY

SECTION

108. Graphical representation of distances.
109. Graphical representation of other number relations.
110. Temperature and other graphs.
111. Graph of an equation.
112. Review exercises.

CHAPTER XVIII

SIMULTANEOUS EQUATIONS

113. Equations Involving Two Unknowns.

It has been shown that a linear equation in two unknowns has an unlimited number of solutions; that is, in the equation $y - 2x = 1$, when $x = 0$, $y = 1$; when $x = 1$, $y = 3$; when $x = 2$, $y = 5$; when $x = 3$, $y = 7$, etc. If a common solution of two such equations should be desired, one might write out the solutions of each until a solution of one which is also a solution of the other should be found. This, however, would be a laborious process, and if there is no common solution, would be an endless one.

The corresponding sets of values of the two unknowns in any equation, for instance, (0, 1), (1, 3), (2, 5), (3, 7), (4, 9), etc., the values of x and y respectively in the equation $y - 2x = 1$, are shown on the graph of the equation (figure 4). These separate sets of values are called co-ordinates, that is, 0 and 1 are co-ordinates, likewise 1 and 3, 2 and 5, 3 and 7, 4 and 9, etc.

We know that the co-ordinates of any point on the graph of an equation, when substituted for the unknowns in the equation, satisfy the equation; thus, the equation $y - 2x = 1$, becomes

$$1 - 2 \cdot 0 = 1, \text{ or } 1 = 1$$

when the co-ordinates (0, 1) are substituted, and it becomes

$$3 - 2 \cdot 1 = 1, \text{ or } 1 = 1,$$

when the co-ordinates (1, 3) are substituted. If the graphs of two equations in two unknowns are drawn, the co-ordinates of their point of intersection satisfy both equations.

ILLUSTRATIVE EXERCISE

Find a common solution of the equations,

$$4x + 3y = 11$$
$$2x + y = 5$$

Solution:

Draw the graphs of the equations.

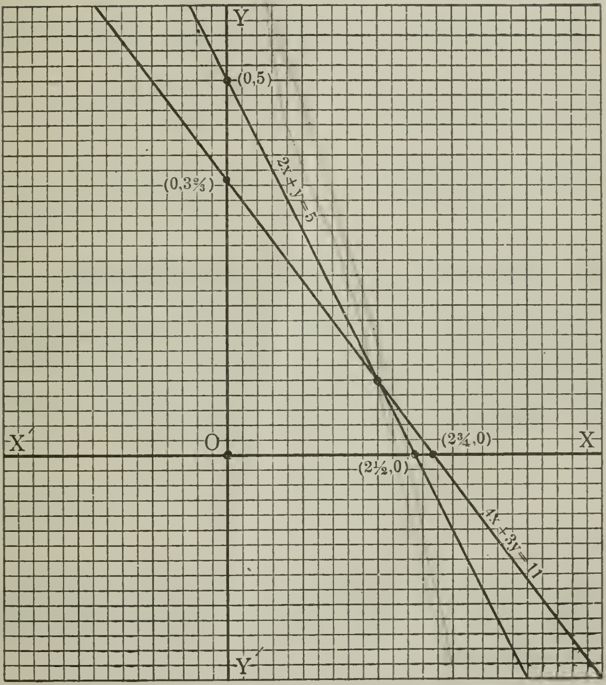


Figure 7. Graph of the equations $4x+3y=11$ and $2x+y=5$.

The graphs are found to intersect at the point (2, 1). (See figure). Therefore, the common solution of the two equations is $x = 2$ and $y = 1$.

EXERCISES

Solve by this method the following pairs of equations:

1. $x + y = 3$
   $x - y = 1$

2. $2x - y = 4$
   $3x + 2y = 13$

3. $x - 3y = 4$
   $3x + 2y = 1$

4. $4x - 3y = 1$
   $2x + y = 13$

5. $3x - 2y = 11$
   $x + 5y = 21$

6. $6x + 2y = 24$
   $5x - y = 22$

7. $3x - 7y = 8$
   $2x + 3y = 13$

8. $5x + 4y = 1$
   $3x - 7y = 10$

9. $9x - 4y = 3$
   $7x + 2y = 33$

10. $2x + y = 7$
    $3y - x = 0$

114. Elimination by Addition and Subtraction.

A shorter method of finding the common solution of two linear equations in two unknowns is to combine the equations in a way that will give an equation in one unknown. Take, for example, the equations,

$$4x + 3y = 11 \tag{1}$$

$$2x + y = 5 \tag{2}$$

Solution:

If we multiply both members of (2) by 3 and subtract from the result (1), the resulting equation will contain only one unknown, x.

$3 \cdot (2)$ $$6x + 3y = 15 \tag{3}$$

$$4x + 3y = 11 \tag{1}$$

$(3) - (1)$ $$2x = 4 \tag{4}$$

$(4) \div 2$ $$x = 2 \tag{5}$$

Substituting 2 for x in either (1) or (2), say in (2),

$$2 \cdot 2 + y = 5 \tag{6}$$

$$y = 1 \tag{7}$$

Check by substituting 2 for x and 1 for y in the original equations.

3·(2) means *multiplying both members of equation* (2) *by* 3. (3) − (1) means *subtracting the members of equation* (1) *from the members of equation* (3). (4)÷2 means *dividing both members of equation* (4) *by* 2.

$$4 \cdot 2 + 3 \cdot 1 = 11$$

$$2 \cdot 2 + 1 \cdot 1 = 5$$

Hence, $x = 2$ and $y = 1$ is a solution that satisfies both equations.

As previously shown, a single equation in two unknowns does not have a solution that may be called *the* solution of the equation, since there is an indefinite number of solutions. Two such equations of the first degree usually have a *single* solution. A solution of this kind is called a **determinate** solution.

Equations (1) and (2) are together called a **system of equations.**

Two or more equations that are satisfied by the same values of the unknowns are said to be **simultaneous.**

A **determinate** system of equations is a system that has only one solution or a limited number of solutions.

The process of getting rid of one of the unknowns is called **elimination.** The method of elimination used above is called **elimination by addition and subtraction.**

ILLUSTRATIVE EXERCISE

Solve the system of equations,

$$5x - 4y = 7 \qquad (1)$$
$$2x + 3y = 12 \qquad (2)$$

Solution:

$3 \cdot (1)$ $\quad 15x - 12y = 21$ $\quad (3)$
$4 \cdot (2)$ $\quad 8x + 12y = 48$ $\quad (4)$
$(4) + (3)$ $\quad 23x = 69$ $\quad (5)$
$(5) \div 23$ $\quad x = 3$ $\quad (6)$

(4) + (3) means *adding the members of equation* 3 *to the members of equation* 4.

Substituting 3 for x in (2),

$$6 + 3y = 12$$

Whence, $\quad 3y = 6$
and $\quad y = 2$

Check: Substituting 3 for x and 2 for y in (1) and (2),

$$5 \cdot 3 - 4 \cdot 2 = 7$$
$$2 \cdot 3 + 3 \cdot 2 = 12$$

We might have eliminated x instead of y, as follows:

$2 \cdot (1)$ $\quad 10x - 8y = 14$ $\quad (3)$
$5 \cdot (2)$ $\quad 10x + 15y = 60$ $\quad (4)$
$(3) - (4)$ $\quad -23y = -46$ $\quad (5)$
Whence, $\quad y = 2$

Substituting 2 for y in (1),

$$5x - 8 = 7$$

Whence, $\quad 5x = 15$
and $\quad x = 3$

Before beginning the solution of a system of equations, one should note which of the unknowns is more easily eliminated.

EXERCISES

Solve the following systems of equations and check your results by substituting in the original equations the values found:

1. $3x - y = 5$
$2x + y = 5$

2. $2x + 3y = 13$
$x + 2y = 8$

3. $x - 4y = -15$
$4x - y = 9$

4. $3x + y = 10$
$2x - y = 5$

5. $3x + 5y = 4$
$9x + 10y = 10$

6. $5x + 4y = 1$
$3x - 7y = 10$

7. $10x + 6y = 7$
$4x + 9y = 5$

8. $9x - 4y = 23$
$7x + 2y = 23$

9. $5x - 3y = 8$
$7y - 6x = 21$

10. $7a + 9b = 71$
$5a + 4b = 41$

11. $4a - 3b = 6$
$5a + 6b = 14$

12. $4s - 7k = 25$
$3s - 5k = 11\frac{3}{7}$

13. $5x + 8v = 19$
$6x - 5v = 15\frac{1}{2}$

14. $.3x + .4y = 4.1$
$.7x - .3y = 3.4$

15. $11s - 5m = 2$
$3s + 8m = 38$

16. $4x + 13y = 42$
$12x - 2y = 3$

17. $5x - 7y = 24$
$8x + 9y = 99$

18. $3x + y = 2y + 24$
$x + 2y = 2x + 2$

Hint. In problem 18, first subtract $2y$ from each member of the first equation and $2x$ from each member of the second equation.

19. $3x + 5 = 7y - 10$
$4x - 2y = 6y - 16$

20. $7m + 2n = 35 - 4m$
$9m + n = 35$

21. $6x + 10y = 7$
$9x - 2y = 8y - 2$

22. $\frac{1}{2}x - 7y = 10$
$\frac{1}{3}x - 4y = 6$

23. $4x - \frac{2}{3}y = 12$
$7x + \frac{1}{2}y = 41$

24. $12m - 7n = -8$
$21m - 4n = \frac{5}{2}$

25. $\frac{3}{4}a - 3b = 3$ 4
$\frac{1}{2}a + 4b = 14$

26. $16x - 11y = 10$
$9x + 13y = 44$

27. $3(x + 2y) = 6y + 3$
$4y - 7x = 9$

28. $6(3x - 4) = 5(x - 1)$
$2(3x + 2y) = 46$

29. $.6x - .5z = 1$
$2.4x + 3.6z = 6$

30. $\frac{3}{5}x - y = 2$
$\frac{2}{3}x + 4y = 26$

115. Elimination by Substitution.

Another useful method of elimination is called **elimination by substitution.** The following problem illustrates the method.

ILLUSTRATIVE EXERCISE

Solve:
$$4x - 3y = 5 \tag{1}$$
$$11x + 8y = 30 \tag{2}$$

Solution:

From (1), we find,
$$4x = 3y + 5 \tag{3}$$

Whence,
$$x = \frac{3y + 5}{4} \tag{4}$$

Substituting in (2) the value of x given in (4),

$$11 \cdot \frac{3y + 5}{4} + 8y = 30 \tag{5}$$

Clearing (5) of fractions

$$11(3y + 5) + 32y = 120 \tag{6}$$

or
$$33y + 55 + 32y = 120 \tag{7}$$

Whence,
$$65y = 65$$

and
$$y = 1$$

We may now substitute the value found for y in either (1) or (2) and find the value of x, but it is usually preferable to substitute this value in (4), thus:

$$x = \frac{3 \cdot 1 + 5}{4} = 2$$

Check:
$$4 \cdot 2 - 3 \cdot 1 = 5$$

and
$$11 \cdot 2 + 8 \cdot 1 = 30$$

Instead of finding the value of x in terms of y, we might have found the value of y in terms of x. Which unknown should be expressed in terms of the other depends upon which will give the easier solution.

EXERCISES

Solve the following systems of equations using elimination by substitution:

1. $x - 3y = 1$
   $2x + 5y = 24$
2. $3x + 4y = 7$
   $2x - y = 1$
3. $3x - 2y = 5$
   $x + 2y = 3$
4. $7x + 10y = 3$
   $x + y = 0$
5. $5a + 9b = 52$
   $2a - 3b = 1$
6. $4x - 2 = 5y$
   $3x - 7y = -8$
7. $8x - 7y = -4$
   $2x + 3y = 13$
8. $3x + 13y = 27$
   $5x - y = -\frac{1}{3}$
9. $12x - 5y = -1$
   $4x + 2y = 7$
10. $.5x + 2y = 8$
    $x + .4y = 7$
11. $\frac{x}{3} + \frac{5y}{6} = -6$
    $3x - 2y = -1$
12. $10x + 3y = 62$
    $\frac{3x}{5y} = \frac{3}{4}$
13. $3x + 2y = 15$
    $5x - 4y = 14$
14. $3x - 7y = 8$
    $2x + 14y = 8$
15. $2x + 11y = 27$
    $4x - 3y = 4$
16. $8s - 5x = 4$
    $7s + 3x = 33$
17. $\frac{3x}{2} - 4y = 2$
    $2x + \frac{5y}{3} = 5\frac{1}{3}$
18. $\frac{4x}{7} - \frac{y}{2} = 5$
    $x + 2y = 26$
19. $4 + 2(2x - y) = 6$
    $10 - 3(x - y) = 4$
20. $3x + y = 2y + 24$
    $x + 2y = 2x + 2$
21. $3(x - y) + 4(x + y) = 21$
    $2(3x + y) - 5x = 16$
22. $2(3a - b) + 12 = 4a + 18$
    $3a - 7b = 4(a - b) + 1$

EXERCISES

For Tilton Page

116. Solve the following systems of equations using the method of elimination which seems better in each case:

1. $x + 2y = 5$
   $2x + y = 7$
2. $2x - u = 7$
   $x + 2u = 11$
3. $5x + y = 38$
   $3x - 2y = 15$
4. $6x + 5y = 12\frac{1}{2}$
   $4x + 8y = 17\frac{2}{3}$

5. $x + 3y = 6y - 9$
$3x + 7y = 37$

6. $16 + 3(x + 2m) = 2(x - m)$
$25 + 7(2x + 6m) = 5 - 2m$

7. $7a + 25b = 46$
$a = 3b$

8. $21r - 17s = -9$
$3r = 2s$

9. $\dfrac{p + 3q}{p - 3q} = 10$
$4p - 4q = 3(p + 1)$

10. $\dfrac{12}{5a + 1} - \dfrac{10}{b + 2} = 0$
$3a + \dfrac{b}{3} = 4$

11. $\dfrac{3}{x - y} + \dfrac{11}{x + y} = \dfrac{66}{x^2 - y^2}$
$\dfrac{3x}{y} + 2 = \dfrac{20}{y}$

12. $\dfrac{1}{x} + \dfrac{2}{y} = 4 \quad (1) \qquad \dfrac{4}{x} + \dfrac{3}{y} = 1 \quad (2)$

Solution:

The system of equations in problem **12** is solved to best advantage by first solving it for $\frac{1}{x}$ and $\frac{1}{y}$, as follows:

4 · (1) $\qquad \dfrac{4}{x} + \dfrac{8}{y} = 16 \qquad (3)$

(3) − (2) $\qquad \dfrac{5}{y} = 15 \qquad (4)$

(4) ÷ 5 $\qquad \dfrac{1}{y} = 3$ and $y = \frac{1}{3} \qquad (5)$

Substituting 3 for $\frac{1}{y}$ in (1), $\qquad \dfrac{1}{x} + 2 \cdot 3 = 4$

Whence, $\qquad \dfrac{1}{x} = -2$ and $x = -\frac{1}{2}$

Check: Substituting $\frac{1}{3}$ for y and $-\frac{1}{2}$ for x in (1) and (2),

$$\frac{1}{-\frac{1}{2}} + \frac{2}{\frac{1}{3}} = -2 + 6 = 4 \quad \text{and} \quad \frac{4}{-\frac{1}{2}} + \frac{3}{\frac{1}{3}} = -8 + 9 = 1$$

13. $\dfrac{m+n}{3}+\dfrac{m-n}{2}=\dfrac{7}{3}$

$2m-3n+\dfrac{n}{2}=\dfrac{7}{2}$

14. $\dfrac{a+b}{a-2}=\dfrac{7}{3}$

$\dfrac{3a-5b}{2a-1}=\dfrac{5}{9}$

15. $\dfrac{8}{3u+2v}=\dfrac{5}{3u-v}$

$\dfrac{6}{u-v}-\dfrac{3}{u+v}=\dfrac{15}{u^2-v^2}$

16. $\dfrac{2}{x}+\dfrac{6}{y}=3$

$\dfrac{4}{x}-\dfrac{3}{y}=1$

17. $\dfrac{5}{x}-\dfrac{3}{y}=7$

$\dfrac{2}{x}+\dfrac{6}{y}=10$

18. $\dfrac{3}{x}=\dfrac{7}{y}-\dfrac{1}{8}$

$\dfrac{6}{y}=\dfrac{4}{x}-\dfrac{1}{2}$

19. $\dfrac{8}{x}+2=\dfrac{5}{y}-1$

$\dfrac{4}{x}=\dfrac{2}{y}-1$

20. $\dfrac{7}{x}-\dfrac{10}{y}=1$

$\dfrac{5}{x}+\dfrac{5}{y}=4\tfrac{1}{6}$

21. $\dfrac{3}{2m}+\dfrac{5}{3n}=8 \qquad \dfrac{1}{2m}+\dfrac{7}{3n}=8$

Hint. Solve problem 21 for $\dfrac{1}{2m}$ and $\dfrac{1}{3n}$

22. $ax+by=a+b$ (1)
$2ax+3by=2a+3b$ (2)

Solution: $2\cdot(1)$ $\quad 2ax+2by=2a+2b$ (3)
$2ax+3by=2a+3b$ (2)
$(2)-(3)$ $\quad by=b$
Whence, $\quad y=1$
Substituting 1 for y in (1), $\quad ax+b=a+b$
Whence, $\quad ax=a$
and $\quad x=1$

Check: Substituting 1 for x and 1 for y in (1) and (2)

$$a\cdot1+b\cdot1=a+b \text{ and}$$
$$2a\cdot1+3b\cdot1=2a+3b$$

23. $mx + 3y = 4m \qquad (1)$

$y - x = 0 \qquad (2)$

Solution:

From (2), $y = x$. Substituting this value for y in (1),

$$mx + 3x = 4m \qquad (3)$$

Factoring the first member of (3),

$$(m + 3)x = 4m \qquad (4)$$

Dividing both members of (4) by the coefficient of x, $(m + 3)$,

$$x = \frac{4m}{m + 3}$$

Since $y = x$,

$$y = \frac{4m}{m + 3}$$

Substituting these values for x and y in (1),

$$m \cdot \frac{4m}{m + 3} + 3 \cdot \frac{4m}{m + 3} = \frac{4m^2}{m + 3} + \frac{12m}{m + 3}$$

$$= \frac{4m^2 + 12m}{m + 3} = \frac{4m(m + 3)}{m + 3} = 4m$$

Substituting in (2),

$$\frac{4m}{m + 3} - \frac{4m}{m + 3} = 0$$

The results check.

Solve the following equations for x and y:

24. $x + 2y = 4m$

$3x - y = 5m$

25. $4x - 5y = 2r$

$2x + 3y = 12r$

26. $11x - 7y = 15b$

$\frac{x}{y} = 2$

27. $$\frac{x}{a} + \frac{y}{b} = 5$$
$$\frac{x}{a} - \frac{y}{b} = 1$$

28. $$\frac{2x}{k} + \frac{5y}{n} = 9$$
$$\frac{3x}{k} - \frac{2y}{n} = 4$$

29. $$\begin{aligned} ax + by &= 2ab \\ 2ax - by &= ab \end{aligned}$$

30. $$\begin{aligned} 5cx + 2dy &= 7cd \\ 3cx + 4dy &= 7cd \end{aligned}$$

31. $$\begin{aligned} 2x + 3y &= 5m - n \\ x - y &= 2n \end{aligned}$$

Hint. Solve problem 27 first for $\frac{x}{a}$ and $\frac{y}{b}$.

117. Problems Involving Two Unknowns.

In the following problems two unknown numbers are to be represented by letters. Since a single equation containing two unknowns has an indefinite number of solutions, it is necessary to have two equations containing the unknowns, and find a common solution. There must be, then, two different statements in the problem, from each of which an equation can be formed containing the letters which represent the numbers whose values are sought.

Solve the following problems, using equations in two unknowns:

1. Find two numbers whose sum is 42 and whose difference is 8.

Solution:

Let $x =$ the greater number
and $y =$ the smaller number
Then $x + y =$ the sum of the two numbers
and $x - y =$ the difference between the two numbers.

The problem states that 42 is the sum of the two numbers and 8 is their difference. Hence,

$$x + y = 42 \qquad (1)$$

and $$x - y = 8 \qquad (2)$$

Solving (1) and (2), we find $x = 25$ and $y = 17$. Hence the two numbers are 25 and 17. Since the sum of 25 and 17 is 42 and their difference is 8, these numbers satisfy the conditions of the problem and are correct.

What statement in the problem gives equation 1? What statement gives equation 2?

2. Find two numbers whose sum is 35 and whose difference is 7.

3. Find two numbers whose sum is 163 and whose difference is 109.

4. Find two numbers whose sum is 4 and whose difference is 36.

5. Find two numbers whose sum is 8 and such that 4 times the smaller exceeds twice the greater by 2.

6. The sum of two numbers is 1 and their difference 15. What are the numbers?

7. A man is at present 3 times as old as his son and 10 years ago he was 5 times as old as his son. What is the age of each?

Let x = the father's age and y the son's. What was the father's age 10 years ago? the son's age 10 years ago?

8. John is 10 years older than Henry. In 5 years he will be twice as old as Henry. What is the age of each?

9. A had twice as much money as B. After spending \$8 apiece, A had 6 times as much as B. How much had each at first?

10. A had 3 times as much money as B. A gave B \$20 and they then had equal amounts. How much had each at first?

Louise Vera Way 6. J. H. C.

Louise
Vera
Dot
Bernice
E. S. J. C.

11. A merchant paid, in one week, \$120 in salaries to his 10 woman clerks and 2 man clerks. The next week, he discharged 2 of his woman clerks and employed another man clerk. For this week, his clerks' salaries amounted to \$117. What was the weekly salary of one woman clerk and of one man clerk?

12. A merchant sold 7 pounds of sugar and 4 pounds of butter to one customer for \$1.74, and 5 pounds of sugar and 3 pounds of butter to another for \$1.29. What was the price of a pound of each?

13. Five cubic feet of mahogany and 2 cubic feet of oak weigh about 476 pounds, and 2 cubic feet of mahogany and 4 cubic feet of oak weigh about 424 pounds. What is the weight of a cubic foot of each?

14. A fruit dealer sold 36 apples and oranges, the apples at 2 cents each and the oranges at 5 cents each. For all he received \$1.08. How many of each did he sell?

Hint. If x represents the number of apples and y the number of oranges, what represents the total number sold? What represents the total amount received?

15. Eight boys and 16 men work in one gang and 4 boys and 12 men in another. The daily payroll for the first gang amounts to \$40, and for the second to \$28. What are the daily wages of a man and of a boy?

16. The price of admission to a moving picture show was 10 cents for adults and 5 cents for children. If there were 350 tickets sold on a certain evening and the receipts were \$25, how many adults and how many children attended the show?

17. The sum of the numerator and denominator of a fraction is 18 and if both the numerator and denominator be increased by 3, the resulting fraction is equal to $\frac{1}{2}$. What is the fraction?

Solution:

Let x = the numerator of the fraction

and y = the denominator of the fraction.

Then, by the conditions of the problem,

$$x + y = 18 \tag{1}$$

and

$$\frac{x+3}{y+3} = \frac{1}{2} \tag{2}$$

Solving equations (1) and (2), we find $x = 5$ and $y = 13$. Hence the fraction is $\frac{5}{13}$.

Check: $$5 + 13 = 18$$

and

$$\frac{x+3}{y+3} = \frac{8}{16} = \frac{1}{2}$$

18. The sum of the numerator and denominator of a fraction is 13. If 2 be added to the denominator, the value of the fraction is $\frac{1}{4}$. What is the fraction?

19. A certain fraction is equal to $\frac{1}{4}$. If 10 be added to the numerator, the value of the resulting fraction is $\frac{1}{2}$. What is the fraction?

20. A fraction is equal to $\frac{1}{2}$. If 4 be added to the numerator, the fraction becomes equal to $\frac{2}{3}$. What is the fraction? Wt.

21. If 4 be added to the numerator and the denominator of a certain fraction, its value becomes $\frac{6}{7}$. If 4 be subtracted from both terms of the fraction, its value becomes $\frac{2}{3}$. Find the fraction.

22. The difference between two numbers is 4, and $\frac{1}{2}$ of the greater exceeds $\frac{1}{3}$ of the smaller by 4. Find the numbers.

23. The quotient of two numbers is 3 and twice the greater diminished by 4 times the smaller is 14. What are the numbers?

24. A's age is $\frac{3}{4}$ of B's age. Twenty years ago A's age was $\frac{4}{7}$ of B's age. What are their ages?

25. A is 9 years older than B. In 10 years B will be $\frac{5}{6}$ as old as A. What are their present ages?

26. The ones' digit of a number is one greater than the tens' digit. The number is 3 greater than 4 times the sum of its digits. What is the number?

If x is the tens' digit and y the ones' digit, what is the number?

27. Find a number of two digits such that the sum of the digits is 5, and if the order of the digits be reversed, the resulting number is 1 less than three times the original number.

28. A number of two digits is equal to 3 times the ones' digit. If the order of the digits be reversed, the resulting number exceeds 3 times the original number by 6. What is the number?

29. A number of two digits is 5 greater than 3 times the sum of its digits. If the order of the digits be reversed, the resulting number exceeds twice the original number by 7. What is the number?

30. In a number of two digits, the tens' digit is two greater than the ones' digit. If the number be divided by the sum of its digits, the quotient is 7. Find the number.

31. A and B are 24 miles apart. If they walk toward each other, they will meet in 3 hours. If they walk in the same direction with A behind B, A will overtake B in 12 hours. How fast does each walk?

Solution:

Let x = the number of miles A walks in one hour,
and y = the number of miles B walks in one hour.

How far does A walk in the 3 hours? B? What represents the sum of the distances walked by A and B in the 3 hours? What does this sum equal?

How far does A walk in the 12 hours? B? What represents the distance which A gains on B in the 12 hours? What does this distance equal?

32. Two trains are 180 miles apart. If they run toward each other they will meet in 3 hours. If they run in the same direction with the faster train behind, they will be together in 9 hours. What is the rate per hour of each?

33. A and B are 38 miles apart. If they walk toward each other, and B starts one hour before A, they will meet 6 hours after A starts. If they walk in the same direction with A behind, and A starts 4 hours before B, they will be together 11 hours after B starts. What is the rate per hour of each?

34. A man started on a journey of 36 miles on a bicycle. After riding 3 hours his bicycle broke down and he walked the remaining distance in 4 hours. With the bicycle repaired, he started back and again broke down after riding $2\frac{1}{2}$ hours and walked the remaining distance in $5\frac{1}{3}$ hours. What was his rate of walking and of riding?

35. A man made a trip of 52 miles, riding 7 hours and walking 5 hours. He made the return trip walking 8 hours and riding 6 hours. If he walked at the same rate and rode at the same rate in both cases, find his rate of walking and of riding.

36. A grocer bought potatoes at \$.90 a bushel and apples at \$1.50 a bushel, paying \$90 for all. He sold the potatoes at \$1.25 a bushel and the apples at \$2.00 a bushel, thereby gaining \$32.50. How many bushels of each did he buy?

37. A fruit dealer bought some bananas at 15 cents a dozen and some at 20 cents a dozen, paying \$4.10 for all. After 2 dozen had spoiled he sold the rest at the rate of 2 for a nickel and made a profit of \$2.50. How many of each did he buy?

38. A man can swim 6 miles an hour downstream and 2 miles an hour upstream. At what rate can he swim in still water and what is the rate of flow of the stream?

Let $x =$ the man's rate of swimming per hour in still water,
and $y =$ the rate per hour at which the stream flows.

What represents the man's rate of swimming downstream? upstream?

39. A man rows 21 miles up a stream in 7 hours and returns to his starting point in 3 hours. How fast can he row in still water and at what rate is the stream flowing?

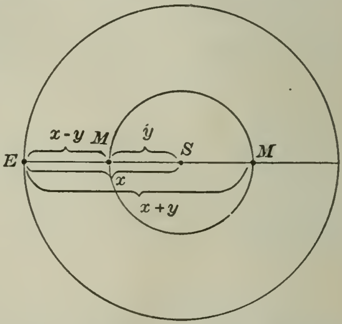
40. An excursion steamer makes a trip of 48 miles up a river in 6 hours and returns in 4 hours. At what rate is the river flowing, and at what rate would the steamer go in still water?

41. Common brass is an alloy of copper and zinc. If copper is worth 15 cents a pound and zinc 4 cents a pound, how many pounds of each must be used to make 25 pounds of common brass worth \$2.76?

42. If the prices of copper and zinc are the same as in the preceding problem, 100 pounds of white brass contain $7.74 worth of material. In what proportion are the metals mixed to make white brass?

43. Bronze is an alloy of copper and tin. The value of the material in a bronze bearing weighing 9 pounds is $1.90. If the copper is worth 20 cents a pound and tin 30 cents a pound, how many pounds of each metal are there in the bearing?

44. Assuming that the planets move in circles with the sun at the center, the greatest distance between the earth and Mercury (when they are on opposite sides of the sun) is 123.8 millions of miles. The shortest distance between them (when they are on the same side of the sun) is 56.8 millions of miles. Mercury being nearer the sun than the earth, find the distance of each from the sun.



45. A and B can dig a ditch in $4\frac{4}{5}$ days. If A works 6 days alone, then B can finish the ditch alone in 4 days. How long will it take each to dig the ditch alone?

Solution:

Let x = the part of the work A can do in one day,
and y = the part of the work B can do in one day.

Then, since $4\frac{4}{5} = \frac{24}{5}$, we have

$$\frac{24}{5}x + \frac{24}{5}y = 1 \quad \text{(the whole work)}$$

and

$$6x + 4y = 1$$

Solving these equations, we find $x = \frac{1}{12}$ and $y = \frac{1}{8}$. Hence, A can dig the ditch in 12 days and B in 8 days.

46. A and B together can do a piece of work in 4 days. A works alone for 3 days and then B alone finishes the work in 6 days. In how many days can each do the work alone?

47. A man has a yearly income of \$490 derived from two investments. One investment is in bonds bearing 4% interest, and the other in a mortgage bearing 7%. If the sum invested in bonds were invested in the mortgage, and, *vice versa*, his income would be \$445. Find the amount of each investment.

48. A and B invest money at 5%, the interest on both sums for one year being \$250. The interest on A's investment will equal B's investment in 5 years. What is the amount of each investment?

118. Systems of Equations Involving Three or More Unknowns.

If there are three unknowns, three independent equations are necessary for a determinate solution. If we have one or two equations in three unknowns, such as,

$$x + 4y + 2z = 10$$

and

$$3x + y + 2z = 14$$

it can be shown, as in the case of one equation in two unknowns, that there is an unlimited number of solutions.

In order to solve a system of three equations in three unknowns, it is necessary to eliminate the same unknown from two pairs of equations. We then have two equations in two unknowns which can be solved as heretofore, and the values found for the two unknowns, substituted in one of the given equations, will give the value of the third unknown.

W. Worthy Walker, Troy Ave & Drawing

ILLUSTRATIVE EXERCISE

Solve the system of equations:

$$x + 2y + 3z = 10 \qquad (1)$$

$$3x + 2y + z = 14 \qquad (2)$$

$$2x + y + 4z = 12 \qquad (3)$$

Solution:

(2) − (1) $2x - 2z = 4$ (4)

(4) ÷ 2 $x - z = 2$ (5)

2 · (3) $4x + 2y + 8z = 24$ (6)

(6) − (2) $x + 7z = 10$ (7)

(7) − (5) $8z = 8$ (8)

(8) ÷ 8 $z = 1$ (9)

Substituting 1 for z in (5) $x - 1 = 2$ (10)

Whence, $x = 3$ (11)

Substituting 1 for z and 3 for x in (3),

$$6 + y + 4 = 12 \qquad (12)$$

Whence, $y = 2$ (13)

Check: Substituting 3 for x, 2 for y, and 1 for z, in (1), (2), and (3),

$$3 + 4 + 3 = 10$$

$$9 + 4 + 1 = 14$$

$$6 + 2 + 4 = 12$$

In solving such equations, the first step is to observe which unknown can be most easily eliminated. Combining the equations in the easiest way will often save considerable work.

In order that a system of equations may have a determinate solution, there must be as many equations as there are unknowns, *i. e.*, four unknowns, four equations, etc.

When we have four equations in four unknowns, we eliminate the same unknown from three pairs of the given equations and then have three equations in three unknowns, which can be solved as above. Elimination by substitution may be used, but it is generally preferable to eliminate by addition and subtraction, as in the example above.

EXERCISES

Solve the following systems of equations:

1. $$\begin{aligned} x+y+z&=5\\ 3x+2y+3z&=13\\ x-3y+2z&=-4 \end{aligned}$$

2. $$\begin{aligned} 5x-y+z&=6\\ 4x+y-z&=3\\ 6x+2y+z&=10 \end{aligned}$$

3. $$\begin{aligned} 4x+2y-z&=15\\ 2x-3y+5z&=19\\ x-3y-4z&=-29 \end{aligned}$$

4. $$\begin{aligned} x+6y+4z&=7\\ 3x-12y-4z&=19\\ 5x+4y-10z&=1 \end{aligned}$$

5. $$\begin{aligned} 2x-5y&=4\\ 3x-2z&=-2\\ 3x-2y-2z&=9 \end{aligned}$$

6. $$\begin{aligned} x+3y+6z&=3\\ 4x-6y+4z&=1\\ 8x+9y-12z&=4 \end{aligned}$$

7. $$\begin{aligned} x+3y&=10\\ 2x-3y+z&=-7\\ 4x+3y-2z&=10 \end{aligned}$$

8. $$\begin{aligned} x+y&=11\\ y+z&=13\\ x+z&=12 \end{aligned}$$

9. $$\begin{aligned} 4x-6y&=z-10\\ 3x-z&=3y-5\\ 2z+4y+5x&=10 \end{aligned}$$

10. $$\begin{aligned} 6m-4n-3p&=37\\ 3m+8n-5p&=-2\\ 9m-2n-p&=4 \end{aligned}$$

11. $$\begin{aligned} x+y+z+v&=7\\ x+y-z+v&=1\\ x-y+z+v&=3\\ x+y+z-v&=5 \end{aligned}$$

12. $$\begin{aligned} a+b&=3\\ c+d&=7\\ b+c&=5\\ 2a+d&=6 \end{aligned}$$

119. Problems Involving Three Unknowns.

Solve the following problems, using three unknowns:

1. The sum of three numbers is 12. The first is 2 more than twice the second and the second is 1 more than twice the third. Find the numbers.

Gs Dr. Dorthy Clothier
Dr. Dwaine Clark
Prof. Toothe Ache in the Hell

Solution:

Let x = the first number
y = the second number
and z = the third number

Then
$$x - 2y = 2 \quad (1)$$
$$y - 2z = 1 \quad (2)$$
and
$$x + y + z = 12 \quad (3)$$

Solving this system of equations, we have $x = 8$, $y = 3$, and $z = 1$. Check:
$$8 - 2 \cdot 3 = 2$$
$$3 - 2 \cdot 1 = 1$$
$$8 + 3 + 1 = 12$$

2. The sum of three numbers is 22. The second is twice the first and the third is 2 more than the second. Find the numbers.

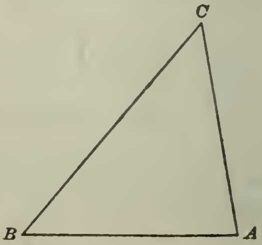
3. Find three numbers such that the sum of 4 times the first, 3 times the second, and twice the third is 5; 3 times the first, minus twice the second, plus 4 times the third is equal to 3; twice the first, plus the second, minus twice the third is equal to 9.

4. In a number consisting of three digits, the sum of the digits is 7. The ones' digit is twice the tens' and the tens' digit is twice the hundreds'. What is the number?

5. In a number consisting of three digits, the ones' digit is twice the hundreds' and the tens' digit is two greater than the hundreds'. The number is equal to 22 times the sum of its digits. What is the number?

Suggestion. If x is the hundreds' digit, y the tens', and z the ones', the number is $100x + 10y + z$.

6. The sum of the three angles of a triangle is 180°. In the triangle ABC, the sum of angle A and angle B is 20° more than 3 times angle C. The sum of $\frac{1}{4}$ of A, $\frac{1}{3}$ of B, and $\frac{1}{2}$ of C is 60°. Find the angles.



[illegible] 687 Roman Age.

7. In a certain triangle, *ABC*, angle *A* is twice angle *B*, and the sum of angles *A* and *B* is 30° more than angle *C*. Find the angles.

8. The perimeter of a triangle, *ABC*, is 17 inches. The sum of the sides *BC* and *AC* is 2 inches greater than the side *AB*. The sum of *AC* and *AB* is 3 times *BC*. Find the lengths of the 3 sides.

9. The perimeter of a triangle is 27 inches. The side *AB* is 3 inches less than the sum of the sides *BC* and *AC*. *AC* is equal to $\frac{1}{2}$ the sum of *AB* and *BC*. Find the lengths of the three sides.

10. In the presidential election of 1912, the total number of electoral votes cast was 531. Mr. Wilson received 5 less than 5 times as many electoral votes as Mr. Roosevelt and Mr. Roosevelt received 11 times as many as Mr. Taft. How many did each receive?

11. The area of Connecticut is 890 square miles greater than twice that of Delaware. Iowa contains 295 square miles less than 8 times the combined areas of Delaware and Connecticut. The sum of the areas of these three states is 63,065 square miles. What is the area of each?

12. A man invested $25,000 in three sums bearing 3, 4, and 5 per cent interest respectively. His yearly income from the three sums was $950. The sum invested at 3% was twice that invested at 5%. Find the amount of each investment.

13. A merchant paid $280 for 100 pairs of shoes, some at $2, some at $3, and some at $4 a pair. By selling the shoes at $3, $4, and $6 a pair respectively, his profit was $120. How many pairs did he buy at each price?

BERNICE

14. Mt. McKinley, Alaska, is 5799 feet higher than Mt. Whitney, California, and 5937 feet higher than Mt. Rainier, Washington. The sum of the heights of the three mountains is 49,164 feet. Find the height of each.

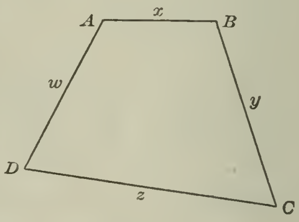
15. The numerator of a fraction whose value is $\frac{4}{3}$ consists of two figures and the denominator of one figure. If 3 is added to the numerator, the value of the fraction is increased by $\frac{1}{3}$, and if 3 is subtracted from the denominator, the value of the fraction is increased by $\frac{2}{3}$. Find the fraction.

Suggestion. Let x be the tens' digit and y the ones' digit of the numerator, and let z be the denominator.

16. In a number consisting of three digits, the hundreds' digit is three times the ones' digit and the tens' digit is twice the ones' digit. The number is 6 more than 15 times the number expressed by the tens' and the ones' digits. Find the number.

17. The perimeter of a triangle is 11 inches more than the side AB. The side BC is $\frac{5}{6}$ of AC and the side AC is $\frac{6}{7}$ of AB. Find the length of each side.

18. The perimeter of a quadrilateral, $ABCD$, is 58 inches. The sum of the sides AB and BC is 4 inches greater than the length of CD. AD is 6 inches shorter than CD and 5 inches longer than AB. Find the length of each side.



19. The sum of the four angles of any quadrilateral is 360°. In $ABCD$, angle A contains 5° less than B and 10° less than twice angle C. The sum of angles C and D is 135°. Find the number of degrees in each angle.

20. A, B, and C together can do a piece of work in $2\frac{2}{3}$ days. A can do the work in $\frac{3}{4}$ of the time that B can and in $\frac{1}{2}$ of the time that C can. In how many days can each do it alone?

REVIEW EXERCISES

120. 1. In how many points can two straight lines intersect?

2. How are two equations of the first degree in two unknowns solved graphically?

3. What is the graph of an equation of the first degree in two unknowns?

4. From your answers to the preceding questions, what do you conclude as to the number of solutions of a system of equations of the first degree in two unknowns?

5. Solve graphically the following system of equations:

$$\begin{aligned} x + y &= 4 \\ 2x + 2y &= 6 \end{aligned}$$

6. If you attempt to solve the system of equations in problem 5 by addition and subtraction, what happens? Such equations are called **incompatible** or **inconsistent** equations. How many solutions does a system of inconsistent equations have?

7. Solve the following system by the method of substitution:

$$\begin{aligned} x &= 2y \\ z &= 3y \\ x + y + z &= 12 \end{aligned}$$

8. Find three different sets of values of x, y, and z that will satisfy the following system of equations:

$$\begin{aligned} x + 2y + 3z &= 12 \\ 3x - y + 3z &= 20 \end{aligned}$$

How many such sets of values can be found?

9. Solve the equation:

$$2x - \frac{2x-3}{5} - 4 = 8 - \frac{2x-3}{3}$$

10. Solve for x and y:

$$x + y = a$$
$$x - y = b$$

11. If oranges cost 30 cents a dozen, what is the cost of x oranges?

12. If hickory nuts are $3.20 a bushel, how many quarts can be bought for y cents? For x dollars?

13. Factor the following:

(1) $(a - 2b)^2 - 4$
(2) $a^3 - 64$
(3) $2x^3 - 3x^2 - 4x + 6$

14. Solve the following equations by factoring:

(1) $c^2 - 5c - 36 = 0$
(2) $x^2 + 4x - 77 = 0$
(3) $3k^2 - 7k - 6 = 0$

SUMMARY

SECTION

113. Solution of linear simultaneous equations in two unknowns by graphs.
114. Elimination by addition and subtraction. Definitions. Determinate solution. System of equations. Simultaneous equations. Determinate system. Elimination. Exercises.
115. Elimination by substitution. Exercises.
116. Miscellaneous exercises.
117. Problems involving two unknowns.
118. Systems of equations involving three or more unknowns.
119. Problems involving three unknowns.
120. Review exercises.

CHAPTER XIX

INVOLUTION AND EVOLUTION

121. Involution is the process of finding a power of a number.

A number used to indicate what power of another number is to be found is called the **index** of the power. In a^3, the ^{3} is the index of the power.

122. Power of a Monomial.

We know that $(a^3)^4$ means that a^3 is to be used 4 times as a factor. Hence, since a^3 means a taken 3 times as a factor, $(a^3)^4$ must mean a taken 4 times 3 times, or 12 times, as a factor. That is,

$$(a^3)^4 = a^3 \cdot a^3 \cdot a^3 \cdot a^3 = aaa \cdot aaa \cdot aaa \cdot aaa = a^{12} = a^{4 \cdot 3}$$

$(a^2)^5$ means that a^2 is to be taken 5 times as a factor. Hence, since a^2 is a taken twice as a factor, $(a^2)^5$ is a taken 10 times as a factor, or

$$(a^2)^5 = a^{10} = a^{5 \cdot 2}$$

If we take the general case, $(a^n)^m$ means that a^n is to be taken m times as a factor, and, since a^n is a taken n times as a factor, we have a taken m times n times as a factor. That is,

$$(\mathbf{a}^{\mathbf{n}})^{\mathbf{m}} = \mathbf{a}^{\mathbf{m} \cdot \mathbf{n}} = \mathbf{a}^{\mathbf{mn}}$$

$$(a^2b^3)^4 = a^2 \cdot a^2 \cdot a^2 \cdot a^2 \cdot b^3 \cdot b^3 \cdot b^3 \cdot b^3 = a^8 \cdot b^{12}$$

It is evident that this process can be applied to any monomial having only positive integral exponents. Hence the following rule.

To raise a monomial to a given power:

Multiply the exponent of each factor of the monomial by the index of the power.

This is called the **index law of involution.**

Note. A numerical factor is usually raised to the required power at once whenever encountered. For example, 4^3 is usually written 64.

EXERCISES

Find the following indicated powers:

1. $(x^2)^2$ x^4
2. $(2^3)^2$ 2^6
3. $(n^2)^3$ n^6
4. $(3^2)^2$ 3^4
5. $(b^2x)^2$ b^4x^2
6. $(a^2x^3)^3$ a^6x^9
7. $(abc^2)^3$ abc^6
8. $(2ax^2)^2$ $2ax^4$
9. $(3a^2c^2)^3$ $3a^6c^6$
10. $(2x^3c^2y)^4$
11. $(ac^2y^3)^4$ ac^8y^{12}
12. $(5^2)^2$ 5^4
13. $(10^2)^3$ 10^6
14. $(2x^2y)^3$
15. $(x^2n^3)^5$ $x^{10}n^{15}$
16. $(3x^4y^5)^3$ $3x^{12}y^{15}$
17. $(a^2x^2y^3)^5$
18. $(4m^3s^2)^3$
19. $(5c^2dx^3)^3$
20. $(6uv^3w^4)^2$
21. $(4r^4s^3d^2)^4$
22. $(\frac{1}{2})^3$ $\frac{1}{6}$
23. $(\frac{3}{4})^2$ $\frac{6}{8}$
24. $(\frac{1}{3}x^2y)^3$
25. $(\frac{2}{3}m^3n^4)^4$
26. $(.2b^3c^4x^2)^5$
27. $(2a^px^n)^3$
    $= 2^3a^{3p}x^{3n}$
28. $(x^3y^n)^3$
29. $(5a^2b^3)^n$
30. $(3r^2c^mx^2)^4 =$

123. Square of a Polynomial.

We have already found the square of a binomial. (See section 72.)

$$(a + b)^2 = a^2 + 2ab + b^2 \qquad (1)$$

Since a and b may represent any two numbers, this formula gives the square of any binomial. For example,

$$\begin{aligned}(5 - 2y)^2 &= 5^2 + 2 \cdot 5 \cdot (-2y) + (-2y)^2 \\ &= 25 - 20y + 4y^2\end{aligned}$$

Here a is 5 and b is $-2y$.

To find the square of a trinomial, $m + n + p$, write it as a binomial, by grouping two of its terms, and then apply (1).

$$\begin{aligned}(m+n+p)^2 &= [(m+n)+p]^2\\ &= (m+n)^2 + 2(m+n)p + p^2\\ &= m^2 + 2mn + n^2 + 2mp + 2np + p^2\\ &= m^2 + n^2 + p^2 + 2mn + 2mp + 2np\end{aligned}$$

Here, $(m+n)$ has been used for a and p for b.

In the same manner it can be shown that

$$\mathbf{(m+n+p+q)^2 =} \tag{2}$$
$$\mathbf{m^2 + n^2 + p^2 + q^2 + 2mn + 2mp + 2mq + 2np + 2nq + 2pq}$$

These examples illustrate the following principle:

The square of a polynomial is equal to the sum of the squares of the terms of the polynomial plus twice the product of each term by each succeeding term.

For example, $(r - 2s - 5)^2$

$$= r^2 + (-2s)^2 + (-5)^2 + 2\cdot r\cdot(-2s) + 2\cdot r\cdot(-5) + 2(-2s)(-5)$$
$$= r^2 + 4s^2 + 25 - 4rs - 10r + 20s$$

EXERCISES

Write by inspection the following squares:

1. $(2x + y)^2$ 4x² + [illegible]
2. $(3a - 2b)^2$ 9a² - 12ab + [illegible]
3. $(5x - 3)^2$ 25x² - 30x + 9
4. $(7a - 4)^2$ 49a² - 56a + [illegible]
5. $(2m + 5p)^2$ 4m² + [illegible]
6. $(x + y + z)^2$ [illegible]
7. $(a + b + 2)^2$ [illegible]
8. $(m - n + 3)^2$ m² + n² - 2mn + 6m - 6n + 9
9. $(x - 3y - 1)^2$ [illegible]
10. $(3k + 2m - p)^2$ 9k² + [illegible]
11. $(s - t - 3r)^2$
12. $(4a + 3c - 5x)^2$ 16a² + [illegible]
13. $(3x^2 + 2x - 4)^2$ [illegible]
14. $(3a^2 - 4a - \frac{1}{2})^2$
15. $(2m - 5xy + .4n)^2$
16. $(x - 2 + y - z)^2$
17. $(a + b + c + d)^2$
18. $(a + b + 3 - k)^2$
19. $(\frac{x}{2} - 2s + 3t)^2$

3k² - 4m + 12 [illegible]

124. Cube of a Binomial.

Show by performing the multiplication that

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \tag{3}$$

If b is replaced by $-b$ in (3),

$$\begin{aligned}(a-b)^3 &= a^3 + 3a^2(-b) + 3a(-b)^2 + (-b)^3\\ &= a^3 - 3a^2b + 3ab^2 - b^3\end{aligned} \tag{4}$$

From (3),

The cube of the sum of two numbers is equal to the cube of the first, plus three times the square of the first times the second, plus three times the first times the square of the second, plus the cube of the second.

From (4),

The cube of the difference between two numbers is equal to the cube of the first, minus three times the square of the first times the second, plus three times the first times the square of the second, minus the cube of the second.

Expand $(2x+3y)^3$.

This problem comes under (3) if a is $2x$ and b is $3y$. Then

$$\begin{aligned}(2x+3y)^3 &= (2x)^3 + 3(2x)^2 3y + 3(2x)(3y)^2 + (3y)^3\\ &= 8x^3 + 36x^2y + 54xy^2 + 27y^3\end{aligned}$$

EXERCISES

Expand the following:

1. $(x+y)^3$
2. $(m-n)^3$
3. $(2h-k)^3$
4. $(x-2a)^3$
5. $(y+3k)^3$
6. $(2y-3)^3$
7. $(3a+2)^3$
8. $(3c-x)^3$
9. $(5m+n)^3$
10. $(3-2x)^3$
11. $(3s+4r)^3$
12. $(2+a+x)^3$

Hint. Write exercise 12 in the form $[(2+a)+x]^3$.

125. Evolution.

If a number is the product of two or more equal factors, each of these equal factors is called a **root** of the number.

If a number is the product of two equal factors, each factor is the **square root** of the number.

Since $64 = 8 \cdot 8$, 8 is the square root of 64.

Since $x^2 + 10x + 25 = (x+5)(x+5)$, $x+5$ is the square root of $x^2 + 10x + 25$.

If a number is the product of three equal factors, each factor is the **cube root** of the number.

Since $125 = 5 \cdot 5 \cdot 5$, 5 is the cube root of 125.

Since $x^6 = x^2 \cdot x^2 \cdot x^2$, x^2 is the cube root of x^6.

If a number is the product of m equal factors, each of these m factors is the **mth root** of the number.

Evolution is the process of finding a root of a number.

The symbol used to indicate a root, *viz.*, $\sqrt{}$, is already known. A number placed above and before this symbol indicates the desired root, and is called the **index** of the root. When no index is written, 2 is understood. Thus $\sqrt{4}$ means the square root of 4; $\sqrt[3]{27}$ means the cube root of 27; $\sqrt[5]{32}$ means the fifth root of 32.

An indicated root is called odd or even according as its index is odd or even.

From the definition, it is evident that finding a root of a number is the inverse of finding a power of a number.

$2^2 = 4$. Hence $\sqrt{4} = 2$. $3^3 = 27$. Hence, $\sqrt[3]{27} = 3$. But,

$(-2)^2 = 4$. Hence, $\sqrt{4} = -2$. $(-3)^3 = -27$. Hence, $\sqrt[3]{-27} = -3$. 1264 27

3

Since an odd power of a number has the same sign as the number, it follows that an odd root of a number has the same sign as the number.

$$\sqrt[3]{-64} = -4, \quad \sqrt[3]{-x^3} = -x, \text{ etc.}$$

An even power of a number (whether positive or negative) is positive. Hence, an even root of a positive number may be positive or negative. As seen above,

$\sqrt{4} = 2$ or -2. This is usually written, $\sqrt{4} = \pm 2$.

$\sqrt{25} = \pm 5$, $\sqrt{16m^2} = \pm 4m$, etc.

There are, then, two square roots of a positive number, differing in sign only.

Since $(a^2b^3c^4)^2 = a^4b^6c^8$, it follows that $\sqrt{a^4b^6c^8} = a^2b^3c^4$.

To obtain a power of a monomial, the exponent of each factor is multiplied by the index of the power. (See section 122.)

Hence, to find a root of a monomial:

Divide the exponent of each factor by the index of the root.

This may be expressed in the following formula,

$$\sqrt[n]{a^m} = a^{\frac{m}{n}}$$

and is called the **index law of evolution.**

EXERCISES

Write two square roots of each of the following numbers:

$36, 81, 121, 49, a^2, b^4, x^2y^2$

Find by inspection the following indicated roots:

1. $\sqrt[3]{64}$ 4

2. $\sqrt[4]{16}$ 2

3. $\sqrt[3]{-125}$ -5

4. $\sqrt[5]{32}$ 2

5. $\sqrt{a^2b^6}$ ab^3

6. $\sqrt[3]{x^3y^6}$ xy^2

7. $\sqrt[4]{x^{12}}$ x^3

8. $\sqrt[5]{k^{10}}$ k^2

9. $\sqrt[3]{m^9}$ m^3

10. $\sqrt{4a^2x^2}$ $2ax$

11. $\sqrt[4]{b^4y^8}$ by^2

12. $\sqrt[3]{27a^3y^3}$ $3ay$

13. $\sqrt{16m^2x^4}$

14. $\sqrt[3]{64c^3x^3}$

15. $\sqrt[3]{-27b^6n^3}$

16. $\sqrt{144c^8x^{10}}$ $12c^4x^5$

17. $\sqrt{36a^4c^6y^8}$ $6a^2c^3y^4$

18. $\sqrt[4]{16m^4n^4y^4}$

19. $\sqrt[5]{32a^5x^{10}}$ $2ax^2$

20. $\sqrt[6]{m^6n^{12}}$ mn^2

21. $\sqrt{x^2(a-b)^2}$ $x(a-b)$

22. $\sqrt[3]{8(x-y)^3}$ $2(x-y)$

23. $\sqrt[3]{\frac{8}{27}}$

Since $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$, it follows that $\sqrt[3]{\frac{8}{27}} = \frac{2}{3}$.

A root of a fraction is found by taking the indicated root of the numerator and of the denominator.

24. $\sqrt{\frac{a^4}{x^4}}$

25. $\sqrt{\frac{4h^2}{9k^2}}$

26. $\sqrt[3]{-\frac{125}{64}}$

27. $\sqrt{\frac{25m^2}{36n^2}}$ $\frac{5m}{6n}$

28. $\sqrt[3]{\frac{b^3}{27c^6}}$ $\frac{b}{3c^2}$

29. $\sqrt[n]{a^{2n}x^{3n}}$

30. $\sqrt[3]{m^3y^{3s}}$

31. $\sqrt[a]{x^{4a}y^a}$

32. $\sqrt{\frac{a^{2n}}{x^{2m}}}$

In the following exercises, if necessary, first separate the numbers under the radicals into factors that are perfect squares.

33. $\sqrt{1764} = \sqrt{4 \cdot 9 \cdot 49} = 2 \cdot 3 \cdot 7 = 42$

34. $\sqrt{784}$

35. $\sqrt{1089}$

36. $\sqrt{1225}$

37. $\sqrt{484}$

38. $\sqrt{676}$

39. $\sqrt{1296}$

40. $\sqrt{3136}$

41. $\sqrt{5184}$

42. $\sqrt{2025}$

Begin Here Work as many of the 80 as you can.

126. Square Root of Polynomials.

It has been shown that

$$(a+b)^2 = a^2 + 2ab + b^2$$

What does $a + b$ represent?

What, then, is $a^2 + 2ab + b^2$?

What can you tell about the terms a^2 and b^2?

What relation does $2ab$ bear to a^2 and b^2?

What is essential in order that a trinomial shall be a perfect square? (See section 73.)

Since $a^2 + 2ab + b^2$ is the square of $a + b$, $a + b$ is by definition the square root of $a^2 + 2ab + b^2$.

B. H.

EXERCISES

Find by inspection the square roots of each of the following:

1. $a^2 + 6a + 9$
2. $x^2 + 12x + 36$
3. $16y^2 - 8y + 1$
4. $4a^2 + 12ax + 9x^2$
5. $9x^2 - 18x + 9$
6. $9y^2 + 16x^2 - 24xy$
7. $y^2 - 10y + 25$
8. $4s^2 + 36sk + 81k^2$
9. B. H. $49z^2 - 28z + 4$
10. $y^2 - 34y + 289$
11. $48ck + 9c^2 + 64k^2$
12. $9x^2 + 42xy + 49y^2$
13. $25c^2 - 90cd + 81d^2$
14. $1 - 22am + 121a^2m^2$
15. $25x^2 - 60xy + 36y^2$
16. $y^4 - 14y^2 + 49$
17. $(a+b)^2 + 2(a+b)c + c^2$
18. $(3+x)^2 - 2(3+x)y + y^2$
19. $9 - 6(x+y) + (x+y)^2$
20. $25 + 10(a+c) + (a+c)^2$

How many square roots has each of the above expressions?

A method of finding the square root of any polynomial B. H. that is a perfect square may be derived from the fact that

$$\sqrt{a^2 + 2ab + b^2} = \pm(a+b)$$

As a rule, only the root $+(a+b)$ is used in what follows.

It is evident that the square root of the first term, a^2, is a, the first term of the root. Subtracting a^2 from $a^2 + 2ab + b^2$ leaves $2ab + b^2$. The second term of the root, b, is obtained by dividing the first term of the remainder, $2ab$, by $2a$. Adding b to $2a$, we have $2a + b$ and this multiplied by b gives $2ab + b^2$. The work may be arranged as follows:

$$\begin{array}{ll}
 & a^2 + 2ab + b^2 \;\big|\, \underline{a + b} \\
 & \underline{a^2 \phantom{{}+2ab+b^2}} \\
\text{Trial divisor, } 2 \cdot a = 2a & \qquad 2ab + b^2 \\
\text{Complete divisor, } 2a + b & \qquad \underline{2ab + b^2} = b\,(2a + b)
\end{array}$$

$2a$ is called the trial divisor and $2a + b$ the complete divisor. In what follows T. D. means trial divisor and C. D. means complete divisor.

ILLUSTRATIVE EXERCISES

1. Find the square root of $9x^2 - 48x + 64$.

Solution:

$$\begin{array}{lr}
 & 9x^2 - 48x + 64 \;\big|\, \underline{3x - 8} \\
 & (3x)^2 = \underline{9x^2 \phantom{{}-48x+64}} \\
\text{T. D.} = 2 \cdot 3x = 6x & -48x + 64 \phantom{{}= -8(6x-8)} \\
\text{C. D.} = \qquad 6x - 8 & \underline{-48x + 64} = -8\,(6x - 8)
\end{array}$$

The square root of the first term ($9x^2$) is $3x$. Subtracting $(3x)^2$, or $9x^2$, the remainder is $-48x + 64$. The trial divisor is $2 \cdot 3x$, or $6x$. Dividing $-48x$ by $6x$ gives the second term of the root, -8. Adding -8 to the trial divisor gives $6x - 8$, the complete divisor. In this problem, the a is $3x$ and b is -8.

2. Find the square root of $a^4 - 10a^3 + 29a^2 - 20a + 4$.

Solution:

$$\begin{array}{l|l|l}
 & a^4 - 10a^3 + 29a^2 - 20a + 4 & \underline{a^2 - 5a + 2} \\
(a^2)^2 = a^4 & \overline{} & \\
2 \cdot a^2 = 2a^2 & -10a^3 + 29a^2 - 20a + 4 & \text{(1st Remainder)} \\
\quad 2a^2 - 5a & -10a^3 + 25a^2 & = -5a(2a^2 - 5a) \\
\hline
2(a^2 - 5a) = 2a^2 - 10a & \quad + 4a^2 - 20a + 4 & \text{(2d Remainder)} \\
\quad 2a^2 - 10a + 2 & \quad + 4a^2 - 20a + 4 & = 2(2a^2 - 10a + 2) \\
\hline
\end{array}$$

The first term of the root is evidently a^2. Subtracting $(a^2)^2$, or a^4, the first term of the remainder is $-10a^3$. This divided by $2a^2$ gives $-5a$, the second term of the root. To complete the divisor, we add $-5a$ to $2a^2$ and obtain $2a^2 - 5a$. We multiply this by $-5a$ and subtract the result from the first remainder, obtaining as a second remainder $4a^2 - 20a + 4$. To find the next term of the root, we regard $a^2 - 5a$ as a single term corresponding to the a of the type problem. The second trial divisor is, then, $2(a^2 - 5a)$, or $2a^2 - 10a$. Dividing the first term of the second remainder, $4a^2$, by the first term of this trial divisor, $2a^2$, gives 2, the third term of the root. Completing the divisor by adding this 2 to the second trial divisor, gives $2a^2 - 10a + 2$. Multiplying this by the third term of the root, 2, and subtracting the result from the second remainder, leaves no remainder. Hence, the square root of the given polynomial is $a^2 - 5a + 2$.

The trial divisor in every case is twice the part of the root already found.

The complete divisor is found by adding the new term of the root to the trial divisor.

The steps in extracting square root are as follows:

1. Arrange the expression according to the ascending or descending powers of some letter.

2. Take the square root of the first term of the expression as the first term of the root.

3. Subtract the square of this term of the root from the given expression.

4. To find the second term of the root, divide the first term of the remainder by twice the first term of the root.

5. Add the second term of the root to the trial divisor (twice the first term of the root), multiply the result by the second term of the root, and subtract the product from the first remainder.

6. To find the third term of the root, take twice the part of the root already found for a trial divisor and divide the first term of the second remainder by the first term of this trial divisor. This will give the third term of the root.

7. Add the third term of the root to the second trial divisor, multiply the sum by the third term of the root and subtract the product from the second remainder.

8. Continue this process until there is no remainder or the desired number of terms has been obtained.

Find the square root of each of the following:

1. $9x^2 + 30x + 25$
2. $x^4 - 6x^3 + 15x^2 - 18x + 9$
3. $y^4 - 4y^3 + 14y^2 - 20y + 25$
4. $x^4 + 6x^3 + 11x^2 + 6x + 1$
5. $x^4 - 2x^3 - x^2 + 2x + 1$

6. $4x^4 - 12x^3 + 25x^2 - 24x + 16$
7. $25m^4 + 80m^3 - 6m^2 - 112m + 49$
8. $a^2 + 6ab + 8ac + 9b^2 + 24bc + 16c^2$
9. $4x^4 - 12x^3 + 29x^2 - 30x + 25$
10. $a^4 - 2a^3 + 2a^2 - a + \frac{1}{4}$
11. $m^4 + m^3 + \frac{9}{4}m^2 + m + 1$
12. $4x^4 + 8x^3 - 4x + 1$
13. $9x^4 - 8x - 8x^2 + 4 + 12x^3$
14. $a^2 + 4b^2 + 9c^2 - 4ab + 6ac - 12bc$
15. $y^4 - 2y^3 + \frac{3}{2}y^2 - \frac{1}{2}y + \frac{1}{16}$

Leave these
not needed

127. Square Root of Arithmetical Numbers.

The square roots of arithmetical numbers can be found by the method given in the preceding section. For example,

ILLUSTRATIVE EXERCISE

Find the square root of 1936.

Solution:

This number lies between 1600 and 2500. Therefore its square root lies between 40 and 50, that is, it is 40 plus a number less than 10. If we represent the ones' digit by b, then

$$\sqrt{1936} = 40 + b$$

and b can be found as in the preceding section.

$$\begin{array}{lr|l}
 & 1936 & \underline{40 + 4 = 44} \\
 40^2 = & \underline{1600} & \\
\text{T. D.} = 2 \cdot 40 = 80 & 336 & \\
\text{C. D.} = 80 + 4 = 84 & \underline{336} = 4 \cdot 84 &
\end{array}$$

Here the trial divisor is $2 \cdot 40$, or 80, and the first remainder is 336. This divided by 80 gives the value of b, *viz.*, 4. Hence,

$$\sqrt{1936} = 40 + 4 = 44$$

The square of a number of one digit contains 1 or 2 digits.

$$2^2 = 4;\ 8^2 = 64;\ \text{etc.}$$

The square of a number of 2 digits contains 3 or 4 digits.

$$10^2 = 100;\ 99^2 = 9801;\ \text{etc.}$$

The square of a number of 3 digits contains 5 or 6 digits.

$$100^2 = 10{,}000;\ 999^2 = 998{,}001;\ \text{etc.}$$

If, then, we begin at the decimal point and point off a number into periods of 2 figures each, the number of periods will equal the number of digits in the integral part of the square root of the given number.

ILLUSTRATIVE EXERCISE

Find the square root of 279,841.

Solution:

Pointing off the number into periods of two figures each beginning at the decimal point, we have 27,98,41. Hence, there are three digits in the required square root and the square root may be extracted as follows:

$$\begin{array}{lrl}
 & 27{,}98{,}41 & \underline{\big|\ 500 + 20 + 9 = 529} \\
 & 500^2 = \underline{25\ 00\ 00} & \\
\text{T. D.} = 2 \cdot 500 = 1000 & 2\ 98\ 41 & \text{(1st Remainder)} \\
\text{C. D.} = 1000 + 20 = 1020 & \underline{2\ 04\ 00} & = 20 \cdot 1020 \\
\text{T. D.} = 2 \cdot 520 = 1040 & 94\ 41 & \text{(2d Remainder)} \\
\text{C. D.} = 1040 + 9 = 1049 & \underline{94\ 41} & = 9 \cdot 1049
\end{array}$$

The square root of 279841 is evidently between 500 and 600 and we take 500 as the first term of the square root ($= a$) and subtract $(500)^2$ ($= a^2$). The first remainder is 29841. The first trial divisor ($2a$) is $2 \cdot 500$, or 1000, and this is contained in the first remainder more than 20 times, but less than 30 times. Hence, we take 20 as the second term of the root (b) and add it to the trial divisor, 1000, to obtain the complete divisor. We multiply the complete divisor, 1020, by 20 and subtracting the result from the first remainder, obtain the second remainder, 9441. The second trial divisor is twice the part of the root already known, *viz.*, $2 \cdot 520$, or 1040. The second remainder, 9441, divided by 1040 gives the third term of the root, 9. Completing the divisor and multiplying by the new term, 9, we obtain 9441, which, subtracted from the second remainder, gives no remainder. Hence,

$$\sqrt{279841} = 500 + 20 + 9 = 529$$

The square of a decimal contains twice as many decimal places as the given decimal.

$$.1^2 = .01 \qquad .9^2 = .81$$

$$.01^2 = .0001 \qquad .99^2 = .9801$$

$$.001^2 = .000001 \qquad .999^2 = .998001$$

To obtain the square root of a decimal, we begin at the decimal point and divide it into periods of two digits each, adding a cipher if necessary, and proceed as in the above problems. If the number has an integral part and a decimal part, we begin at the decimal point and divide it in each direction into periods of two digits each.

ILLUSTRATIVE EXERCISE

Find the square root of 4.7089.

Solution:	4.70,89	$2 + .1 + .07 = 2.17$
	$2^2 = 4$	
1st T. D. $= 2 \cdot 2 = 4$	.70 89	
1st C. D. $= 4 + .1 = 4.1$	.41 00	$= .1\,(4.1)$
2d T. D. $= 2 \cdot 2.1 = 4.2$	.29 89	
2d C. D. $= 4.2 + .07 = 4.27$	.29 89	$= .07 \cdot (4.27)$

Hence, $\sqrt{4.7089} = 2 + .1 + .07 = 2.17$.

If the student notes at the beginning the number of the periods in the integral part or in the decimal part, and knows, thereby, where the decimal point is to be placed in the result, it will be unnecessary to keep track of the decimal point as was done in the above problem.

EXERCISES

Find the square root of each of the following:

1. 729
2. 1225
3. 1681
4. 6889
5. 16129
6. 3844
7. 108241
8. 522729
9. 547.56
10. 81.5409
11. .2401
12. .165649
13. 4076.8225
14. .00459684
15. 16.040025

16. Find the square root of 3 to 3 decimal places.

Solution:	3.00,00,00	$1000 + 700 + 30 + 2$
	1 00 00 00	
$2 \cdot 1000 = 2000$	2 00 00 00	
$2000 + 700 = 2700$	1 89 00 00	
$2 \cdot 1700 = 3400$	11 00 00	
$3400 + 30 = 3430$	10 29 00	
$2 \cdot 1730 = 3460$	71 00	
$3460 + 2 = 3462$	69 24	

Hence, $\sqrt{3} = 1.732+$.

By noticing at the start that the square root of 3 is 1 plus a decimal, it is not necessary to keep track of the decimal point throughout the process.

Find the square root of each of the following correct to 3 decimal places:

17. 2
18. 5
19. 6
20. 7
21. 10
22. 11
23. 13
24. 14
25. 15
26. 17
27. 19
28. 21
29. 7.5
30. $12\frac{3}{4}$
31. 4.05
32. $\frac{7}{8}$ =.875
33. $\frac{2}{3}$
34. $\frac{2}{7}$
35. $\frac{5}{6}$

128. Square Root of a Fraction.

The square root of a fraction is found by extracting the square root of the numerator and of the denominator.

Thus, $\frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$. Hence, $\sqrt{\frac{4}{9}} = \frac{2}{3}$.

When the denominator is not a perfect square, the approximate root may be found in the same way, but the process is unnecessarily laborious. To find the square root of $\frac{3}{7}$ by this process involves finding the square root of both 3 and 7 and dividing by a number consisting of several digits. Thus,

$$\sqrt{\frac{3}{7}} = \frac{\sqrt{3}}{\sqrt{7}} = \frac{1.732}{2.645} = .6546$$

Find in this way the square root of $\frac{5}{11}$; of $\frac{3}{13}$.

This can be materially shortened in either of two ways:

1. By reducing the fraction to a decimal, as suggested in problems 29 and 32 above.

2. By multiplying numerator and denominator of the fraction by a number that will make the denominator a perfect square. For example,

$$\sqrt{\frac{3}{7}} = \sqrt{\frac{21}{7^2}} = \sqrt{\frac{21}{49}} = \frac{1}{7}\sqrt{21}$$

3/5 $\frac{2}{7}$ 17

We shall now have to find the approximate root of only one number and divide by a number of one digit, *viz.*, 7.

$$\sqrt{\tfrac{3}{7}} = \tfrac{1}{7}\sqrt{21} = \frac{4.5825}{7} = .6546$$

Which is the simpler form, $\sqrt{\tfrac{3}{7}}$ or $\tfrac{1}{7}\sqrt{21}$? Why?

Show that $\sqrt{\tfrac{2}{3}} = \tfrac{1}{3}\sqrt{6}$, and that $\sqrt{\tfrac{1}{2}} = \tfrac{1}{2}\sqrt{2}$.

Which is the simpler form, $\sqrt{\tfrac{2}{3}}$ or $\tfrac{1}{3}\sqrt{6}$? $\sqrt{\tfrac{1}{2}}$ or $\tfrac{1}{2}\sqrt{2}$?

Find the square root of $\tfrac{2}{5}$ correct to 3 decimal places.

$$\sqrt{\tfrac{2}{5}} = \sqrt{\tfrac{10}{25}} = \tfrac{1}{5}\sqrt{10} = \frac{3.162}{5} = .632$$

EXERCISES

Find by this method the square root of each of the following correct to 3 decimal places:

1. $\frac{2}{3}$
2. $\frac{5}{6}$
3. $\frac{3}{8}$
4. $\frac{5}{7}$
5. $\frac{2}{11}$
6. $\frac{3}{5}$
7. $\frac{8}{13}$
8. $\frac{4}{7}$
9. $\frac{3}{13}$
10. $\frac{6}{17}$
11. $\frac{7}{3}$
12. $2\frac{1}{5} = \frac{11}{5}$

Find the value of the following correct to 3 decimal places:

13. $\dfrac{1}{\sqrt{6}} = \sqrt{\tfrac{1}{6}}$
14. $\dfrac{10}{\sqrt{5}} = 10 \cdot \sqrt{\tfrac{1}{5}}$
15. $\dfrac{3}{\sqrt{11}}$
16. $\dfrac{4}{\sqrt{15}}$
17. $\dfrac{6}{\sqrt{14}}$
18. $\dfrac{8}{\sqrt{12}}$

129. Square Root of a Number Having a Factor Which is a Perfect Square.

If the number whose square root is desired contains a factor that is a perfect square, it is usually advisable to take the square root of this factor and find the approximate square root of the remaining factor.

Find the square root of 12.

$$\sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2 \cdot \sqrt{3}$$

Since $\sqrt{3} = 1.732$, $\sqrt{12} = 2(1.732) = 3.464$.

EXERCISES

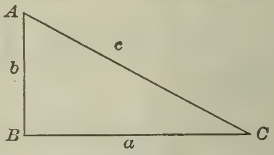
Find the square root of each of the following:

1. 8
2. 20
3. 24
4. 108
5. 48
6. 50
7. 45
8. 80
9. 98
10. 96
11. 63
12. 84
13. 200
14. 44
15. 125
16. 117
17. 252
18. 375
19. 360
20. 288

From the above it is evident that a radical expressing a square root is in its simplest form when the number under the radical sign is an integer and does not contain a factor which is a perfect square.

130. Some Applications of Square Root.

In geometry it is shown that the square on the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides. Let *ABC* be a right triangle. Then, if we represent the lengths of the sides, as indicated in the figure, by a, b, and c, we have



$$a^2 + b^2 = c^2 \tag{1}$$

Subtracting b^2 from each member of (1),

$$a^2 = c^2 - b^2 \tag{2}$$

Subtracting a^2 from each member of (1),

$$b^2 = c^2 - a^2 \tag{3}$$

Extracting the square root of each member of (1), (2), and (3), respectively,

$$c = \sqrt{a^2 + b^2} \qquad (4)$$

$$a = \sqrt{c^2 - b^2} \qquad (5)$$

$$b = \sqrt{c^2 - a^2} \qquad (6)$$

The sides a and b are called the legs of the triangle and c is the hypotenuse.

ILLUSTRATIVE EXERCISE

The legs of a right triangle are 12 feet and 5 feet respectively. Find the length of the hypotenuse.

Solution: a is 12 and b is 5. Hence, from (4),

$$c = \sqrt{12^2 + 5^2} = \sqrt{169} = 13$$

The hypotenuse c is 13 feet long.

EXERCISES

Find the length of the hypotenuse of each of the following right triangles:

1. $a = 40$ ft.; $b = 9$ ft.

2. $a = 8$ in.; $b = 15$ in.

3. $a = 16$ ft.; $b = 12$ ft.

4. $a = 25$ ft.; $b = 17$ ft.

B. H. Moon

5. The hypotenuse of a right triangle is 80 rods and one leg is 40 rods. Find the length of the other leg.

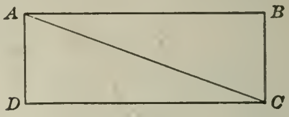
6. The hypotenuse of a right triangle is 100 meters and one leg is 65 meters. Find the length of the other leg.

7. The hypotenuse of a right triangle is 30 feet and one leg is 20 feet. Find the length of the other leg.

8. A ladder 16 feet long will just reach the top of a wall when one end of the ladder is placed on the ground 8 feet from the bottom of the wall. What is the height of the wall?

9. The top of a tree broken off by the wind touches the ground 20 feet from the foot of the tree. The stump is 15 feet high. How high was the tree before it was broken?

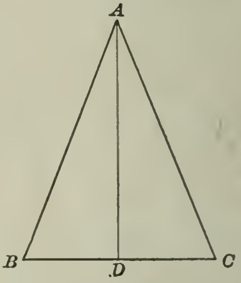
10. In the rectangle $ABCD$, AB is 24 inches and BC is 9 inches. Find the length of the diagonal AC.



11. The width of a rectangle is 26 feet and the diagonal of the rectangle is 40 feet long. Find the length of the rectangle.

12. A rectangular field is 80 rods long and 40 rods wide. A man at one corner of the field wishes to go to the diagonally opposite corner. If he can walk 12 rods a minute around the field, and 10 rods a minute across the field, how much time will he save by going across the field?

An isosceles triangle has two equal sides. ABC is an isosceles triangle, AB and AC being the equal sides and BC the base. The line AD, drawn from A perpendicular to the base, meets the base at its middle point and is called the altitude of the triangle.



13. Find the altitude of an isosceles triangle whose base is 10 inches and one of whose equal sides is 12 inches.

Solution:

Let ABC be the triangle. The altitude AD divides the triangle into two equal right triangles, ABD and ACD. AC is 12 inches; DC is $\frac{1}{2}$ of 10 inches, or 5 inches. Let the altitude AD be x. Then,

$$x = \sqrt{12^2 - 5^2} = \sqrt{144 - 25} = \sqrt{119} = 10.9$$

Hence, the altitude is 10.9 inches.

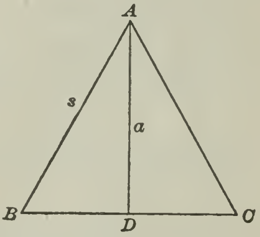
14. Find the altitude of an isosceles triangle one of whose equal sides is 25 feet and whose base is 30 feet.

15. Find the base of an isosceles triangle whose altitude is 5 feet and one of whose equal sides is 13 feet.

16. The ridge pole of a barn is 10 feet above the lower edges of the roof. If the width of the barn between these edges of the roof is 36 feet, how long must the rafters be?

An equilateral triangle is a triangle all of whose sides are equal. It is evident that an equilateral triangle is also isosceles.

Let ABC be an equilateral triangle and s represent the length of each side. Let the length of the altitude AD be a. Then BD is $\frac{s}{2}$. Since ABD is a right triangle, we have



$$a = \sqrt{s^2 - \left(\frac{s}{2}\right)^2} = \sqrt{s^2 - \frac{s^2}{4}}$$

$$= \sqrt{\frac{3s^2}{4}} = \sqrt{\frac{s^2}{4}} \cdot \sqrt{3} = \frac{s}{2} \cdot \sqrt{3}.$$

That is, the altitude of an equilateral triangle is equal to half the length of one side multiplied by $\sqrt{3}$.

17. Find the altitude of an equilateral triangle one of whose equal sides is 12 feet.

Solution:

Let a represent the altitude and s the length of one side. Then,

$$a = \frac{s}{2} \cdot \sqrt{3} = \frac{12}{2} \cdot \sqrt{3} = 6 \cdot (1.732) = 10.392 \text{ (feet)}$$

This problem may be solved without using the formula.

In the figure above, $AB = s = 12$; $BD = \frac{s}{2} = 6$. Hence, since ABD is a right triangle,

$$AD = a = \sqrt{12^2 - 6^2} = \sqrt{144 - 36} = \sqrt{108}$$

$$= 6 \cdot \sqrt{3} = 10.392$$

Find the altitude of an equilateral triangle one side of which is:

18. 16 inches

19. 30 meters

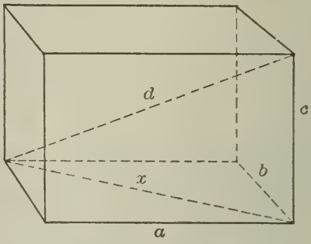
20. 25 rods

21. 18 centimeters

22. 27 feet

If a, b, and c are the length, width, and height, respectively, of a rectangular box, and x is the diagonal of the bottom of the box, then (see accompanying figure),

$$x^2 = a^2 + b^2$$



If d is the diagonal of the box drawn from one lower corner to the diagonally opposite upper corner,

$$d^2 = x^2 + c^2 = a^2 + b^2 + c^2 \qquad (1)$$

Solving (1) for d, we have

$$d = \sqrt{a^2 + b^2 + c^2} \qquad (2)$$

23. Solve (1) for a, b, and c.

$D^2 = a^2 + b^2 + c^2$

24. Find the length of the diagonal of a box 9 feet by 6 feet by 5 feet.

Solution:

Let d represent the length of the diagonal. Then,

$$d = \sqrt{9^2 + 6^2 + 5^2} = \sqrt{81 + 36 + 25}$$
$$= \sqrt{142} = 11.92 \text{ (feet)}$$

25. Find the length of the diagonal of a box car 40 feet long, 8 feet wide, and 9 feet high, correct to 2 decimal places.

1600

26. A cube is a box-shaped solid whose length, breadth, and height are equal. If x represents the length of an edge of a cube, show that the length of the diagonal is $x \cdot \sqrt{3}$.

Find the length of the diagonal of a cube an edge of which is:

27. 15 inches

28. 8 feet

29. 4 feet 6 inches

30. The diagonal of a cubical bin is 20 feet. Find the length.

Hint. Let x represent the length. Then, from problem 26, the length of the diagonal is $x \cdot \sqrt{3}$. Hence, $x \quad = ?$

31. A rectangular coal bin is 12 feet long and 9 feet high and its diagonal is 17 feet long. Find the width.

The area of any triangle is equal to one-half the product of its base and altitude. Any side of a triangle may be regarded as the base and the altitude is the perpendicular from the opposite vertex to the base. In a right triangle, if one leg is taken as the base, the other leg is the altitude.

32. The hypotenuse of a right triangle is 17 feet and one leg is 15 feet. Find the area of the triangle.

Hint. First find the other leg. One leg is the base of the triangle and the other the altitude.

33. The two legs of a right triangle are equal and the area is 72 sq. in. Find the legs and the hypotenuse.

Let x be the length of one leg.

34. The base of an isosceles triangle is 20 inches and one of the equal sides is 16 inches. Find the area.

35. The area of an isosceles triangle is 60 sq. cm. and the altitude is 5 cm. Find the length of one of the equal sides.

36. The diagonal of a square is 20 inches. Find the area.

37. A rectangle is twice as long as wide and the area is 648 sq. in. Find the length, width, and diagonal.

38. If s is one side of an equilateral triangle, prove that the area of the triangle is $\frac{s^2}{4}\sqrt{3}$.

39. Find the area of an equilateral triangle one side of which is 10 feet.

REVIEW EXERCISES

131. **1.** Express in words the index law of involution, *viz.*, $(a^m)^n = a^{mn}$.

2. Find the following indicated powers:

(a) $(3acx)^3$
(b) $(2x^2m^3)^2$
(c) $(12m^2p^3y^4)^3$
(d) $(-25cd^4)^2$
(e) $(-3ax^3z^4)^3$
(f) $\left(\frac{3m^2}{2n^3}\right)^4$

3. Write by inspection the following squares:

(a) $(4x - 11c^2)^2$
(b) $(2x + \frac{3}{2})^2$
(c) $(2a - 5bc + 8)^2$
(d) $(a - 3x + 2m - n)^2$

4. One side of a square field is $a + b$ rods long. How many acres are there in the field?

5. The side of one square is 3 times that of another square. Compare their areas.

6. Determine which of the following are perfect squares and find the square root of each of the perfect squares by inspection:

(a) $4r^2 - 12rs + 9s^2$
(b) $z^2r^4 + 5zr^2 + 25$
(c) $225m^4 - 120m^2c + 16c^2$
(d) $16m^2n^4 - 56mn^2 + 49$
(e) $25x^6y^6 - 10x^3y^3 + z^3$
(f) $36y^2 - 132yz^3 + 121z^6$

7. Add to each of the following such a number that the resulting expression shall be a perfect square:

(a) $m^2 + 4m$
(b) $9r^2 - 6r$
(c) $4x^2 - 24x$
(d) $25p^4 + 20p^2$
(e) $49c^2 - 42c$
(f) $16z^2 + 48z$

8. Find the square root of $16a^4 - 24a^3 + 25a^2 - 12a + 4$.

9. One leg of a right triangle is 7 inches longer than the other and the hypotenuse is 2 inches longer than the longer leg. Find the length of each side of the triangle.

10. Find the square root of each of the following to two decimal places:

(a) $\frac{1}{7}$ (b) $\frac{4}{5}$ (c) $\frac{3}{11}$

11. Find the following indicated roots:

(a) $\sqrt{25a^4b^6}$ (b) $\sqrt[3]{-64x^6y^3z^3}$ (c) $\sqrt[4]{625m^4n^8p^8}$

12. Find the width of a rectangle whose length is 20 feet and whose diagonal is 25 feet.

13. Compute to two decimal places the length of the diagonal of a room 10 feet by 8 feet by 9 feet.

SUMMARY

SECTION

121. Definition of involution.
122. Power of a monomial. Index law of involution.
123. Square of a polynomial.
124. Cube of a binomial.
125. Definition of evolution, root, square root, cube root, mth root. Root of a monomial.
126. Square root of polynomials.
127. Square root of arithmetical numbers.
128. Square root of a fraction.
129. Square root of a number having a factor which is a perfect square.
130. Some applications of square root.
131. Review exercises.

CHAPTER XX

QUADRATIC EQUATIONS

132. A **quadratic** equation is an equation of the second degree, *e. g.*, $3x^2 - 5x + 7 = 0$.

Every quadratic equation may be reduced to the form,

$$\mathbf{ax^2 + bx + c = 0}$$

where a may have any value other than 0, and b and c any values.

Quadratic equations are sometimes classified as **complete** and **incomplete.**

A **complete quadratic equation** contains both the first and the second powers of the unknown.

$3x^2 - 5x = 7$ is a complete quadratic equation.

An **incomplete quadratic equation** contains the second and no other power of the unknown.

$5x^2 - 17 = 28$ is an incomplete quadratic equation.

A complete quadratic equation is sometimes called an **affected** quadratic and an incomplete quadratic equation is called a **pure** quadratic equation.

133. Solution of Quadratic Equations by Factoring. The method of solving equations by factoring has already been given. If the factors of quadratic equations can be found readily, this method of solving them is the most expeditious.

ILLUSTRATIVE EXERCISE

Solve the equation $x^2 - 14x + 24 = 0$.

Solution: Factoring the first member

$$(x-2)(x-12) = 0$$

Setting each factor equal to zero,

$x - 2 = 0$, whence $x = 2$

and $x - 12 = 0$, whence $x = 12$

Checking for $x = 2$: $4 - 28 + 24 = 0$

Checking for $x = 12$: $144 - 168 + 24 = 0$

EXERCISES

Solve the following equations by factoring:

1. $x^2 - 3x + 2 = 0$
2. $x^2 - 13x + 40 = 0$
3. $x^2 - 3x - 40 = 0$
4. $x^2 + 11x + 30 = 0$
5. $x^2 - x = 30$
6. $a^2 + a - 42 = 0$
7. $x^2 - 49 = 0$
8. $y^2 - 5y - 36 = 0$
9. $x^2 - 10x - 39 = 0$
10. $m^2 - m = 90$ 90 -10
11. $p^2 - 3p = 0$
12. $x^2 + 10x = 0$
13. $k^2 - 2k = 120$
14. $r^2 - 7r - 60 = 0$
15. $x^2 + 21x + 98 = 0$
16. $2x^2 - 11x + 15 = 0$
17. $7x^2 - 3x - 4 = 0$
18. $5x^2 - 11x + 2 = 0$
19. $3x^2 - x - 14 = 0$
20. $21t^2 - 19t - 2 = 0$
21. $7x^2 - 41x = 6$
22. $8m^2 - 2m = 15$
23. $10y^2 - 19y - 15 = 0$
24. $9x^2 - 64 = 0$
25. $16x^2 - 50x = 21$
26. $15v^2 - 26v + 8 = 0$
27. $x^2 + (a+2)x + 2a = 0$
28. $x^2 + (a+b)x + ab = 0$
29. $m^2 - (a+b)m + ab = 0$
30. $c^2 - (a-5)c - 5a = 0$
31. $y^2 - 3y + cy - 3c = 0$
32. $z^2 - az - 2bz + 2ab = 0$
33. $ax^2 - (ac+k)x + ck = 0$

134. Solution of quadratic equations by completing the square.

The first member of many quadratic equations can not be readily factored even when reduced to the form $ax^2 + bx + c = 0$. A method of solution must be found which will apply to all quadratic equations. To find this method it is necessary to recall what conditions are necessary in order that a trinomial may be a perfect square.

If a trinomial is a perfect square, what must two of the terms be? What relation must the remaining term bear to these two terms? (See section 73.)

EXERCISES

Determine which of the following are perfect squares:

1. $a^2 + 6ab + 9b^2$
2. $4x^2 - 12xy + 9y^2$
3. $9x^2 - 24x + 14$
4. $49x^2 + 7x + 1$
5. $36k^2 + 36k + 9$
6. $4m^2 - 20m + 25$
7. $16x^2 - 56x + 7$
8. $9y^2 + 42y + 49$
9. $81 + 72x + 16x^2$
10. $100x^2 - 60x + 9$
11. $121a^2 - 44a - 4$
12. $25x^2 - 70x + 49$

ILLUSTRATIVE EXERCISE

1. What number must be added to $9x^2 + 4$ to make it a perfect trinomial square?

Solution:

Since the two terms given are squares, the number to be added must be twice the product of their square roots, *viz.*,

$$2 \cdot \sqrt{9x^2} \cdot \sqrt{4} = 2 \cdot 3x \cdot 2 = 12x$$

Since a number has two square roots, we may also have $-12x$

if we take the negative value of one of the square roots, *e. g.*,

$$2 \cdot \sqrt{9x^2} \cdot \sqrt{4} = 2 \cdot 3x \cdot (-2) = -12x$$

Hence, we may add $+12x$ or $-12x$ and obtain

$$9x^2 \pm 12x + 4 = (3x \pm 2)^2$$

It will be observed that taking the negative roots of both radicals gives $+12x$.

2. Add a number to $4x^2 + 20x$ that will make it a perfect trinomial square.

Solution:

Since $20x$ is twice the product of the square roots of the other two terms, that product is $10x$. Hence, $10x \div \sqrt{4x^2}$ will give the square root of the number to be added.

$$10x \div \sqrt{4x^2} = 10x \div (\pm 2x) = \pm 5$$

We must add $(\pm 5)^2$, or 25, to $4x^2 + 20x$ to obtain a perfect square, which is $4x^2 + 20x + 25$.

EXERCISES

Add to each of the following expressions the number necessary to make it a perfect trinomial square:

1. $x^2 + 4$
2. $x^2 + 9$
3. $x^2 + 25$
4. $x^2 + 81$
5. $m^2 + 4m$
6. $d^2 + 10d$
7. $4 + 20x$
8. $64u^2 - 48u$
9. $4z^2 + 44z$
10. $4c^2 + 12c$
11. $9y^2 + 6y$
12. $25a^2 + 4$
13. $m^2 - 6m$
14. $16k^2 - 8k$
15. $9x^2 - 12x$
16. $x^2 + 2ax$
17. $4x^2 - 12cx$
18. $9c^2 - 30ac$
19. $100s^2 + 20s$
20. $49t^2 - 21t$
21. $64s^2 + 25$
22. $36y^2 - 60y$
23. $81x^2 + 36x$
24. $9k^2 + 49$
25. $a^2x^2 + 2axy$
26. $c^2y^2 - 6cy$
27. $x^2 - .4x$

The solution of a quadratic equation by **completing the square** is illustrated by the following problems:

ILLUSTRATIVE EXERCISES

1. Solve $x^2 + 6x = 27$

Solution:

Completing the square by adding 9 to each member,

$$x^2 + 6x + 9 = 36$$

Extracting the square root of each member,

$$x + 3 = \pm \sqrt{36} = \pm 6$$

Whence, by the subtraction axiom,

$$x = 3 \text{ or } -9$$

2. Solve $25x^2 - 70x = -40$

Solution:

Completing the square,

$$25x^2 - 70x + 49 = -40 + 49 = 9$$

Extracting the square root of each member,

$$5x - 7 = \pm 3$$
$$5x = 10 \text{ or } 4$$
$$x = 2 \text{ or } \tfrac{4}{5}$$

Checking for $x = 2$: $100 - 140 = -40$

Checking for $x = \frac{4}{5}$: $16 - 56 = -40$

The completion of the square should be tested by noticing whether the term of first degree in the unknown is twice the product of the square roots of the other two terms. For example, in the last problem, $-70x$ is twice the product of the square roots of the other two terms. That is,

$$2 \cdot \sqrt{25x^2} \cdot \sqrt{49} = 2 \cdot 5x \cdot (\pm 7) = \pm 70x$$

EXERCISES

Find the roots of the following equations, leaving those containing radicals in the radical form:

1. $x^2 + 4x = 21$
2. $x^2 - 10x = -16$
3. $x^2 + 20x = -96$
4. $4c^2 - 4c = 3$
5. $x^2 + 16x = 105$
6. $9v^2 - 6v = 24$
7. $a^2 + 4a = -2$
8. $x^2 + 3x + 3 = 0$
9. $9r^2 - 12r = 165$
10. $z^2 - 2az = 3a^2$
11. $4v^2 + 20v = 24$
12. $16x^2 - 8x - 120 = 0$
13. $x^2 - 3x - 5 = 0$
14. $x^2 + 7x - 18 = 0$
15. $25m^2 - 20m = 32$
16. $16y^2 - 40y = 50$
17. $s^2 - 18s = -17$
18. $9n^2 - 60n = 25$
19. $100m^2 - 80m = 105$
20. $36y^2 + 24y = 60$
21. $a^2x^2 + 2abx = -b^2$
22. $4a^2x^2 + 4abx = -4ac$
23. $c^2y^2 - 8cy = 9$
24. $9c^2y^2 - 6bcy = 3b^2$

When the coefficient of the second power of the unknown is not a perfect square, it is necessary to multiply or divide both members of the equation by a number that will make this coefficient a perfect square.

ILLUSTRATIVE EXERCISE

Solve the equation $3x^2 - 7x = 6$

The first term may be made a perfect square by (1) dividing both members of the equation by 3, (2) by multiplying both members of the equation by 3, or (3) multiplying both members of the equation by $4 \cdot 3$, or 12. First solution:

Dividing both members by 3, $x^2 - \frac{7}{3}x = 2$

Completing the square, $x^2 - \frac{7}{3}x + \frac{49}{36} = 2 + \frac{49}{36} = \frac{121}{36}$

Extracting the square root of both members,

$$x - \frac{7}{6} = \pm \frac{11}{6}$$

Whence, $x = \frac{7}{6} \pm \frac{11}{6} = 3 \text{ or } -\frac{2}{3}$

Second solution:

Multiplying both members of (1) by 3, $9x^2 - 21x = 18$

Completing the square, $9x^2 - 21x + \frac{49}{4} = 18 + \frac{49}{4} = \frac{121}{4}$

Extracting the square root of both members,

$$3x - \frac{7}{2} = \pm \frac{11}{2}$$

Whence, $3x = \frac{7}{2} \pm \frac{11}{2} = 9 \text{ or } -2$

and $x = 3 \text{ or } -\frac{2}{3}$

Third solution:

Multiplying both members of (1) by 4 times the coefficient of x^2, $36x^2 - 84x = 72$

Completing the square, $36x^2 - 84x + 49 = 72 + 49 = 121$

Extracting the square root of both members,

$$6x - 7 = \pm 11$$

Whence, $6x = 7 \pm 11 = 18 \text{ or } -4$

and $x = 3 \text{ or } -\frac{2}{3}$

The three solutions differ only in the manner in which the term containing x^2 is made a perfect square. The square is completed in each case by adding to each member the square of the quotient obtained by dividing the second term by twice the square root of the first term.

Dividing by the coefficient of x^2 keeps the numbers small, but usually gives fractions. Multiplying by the coefficient of x^2 gives fractions when the coefficient of x is an odd number. Multiplying by 4 times the coefficient of x^2 gives larger numbers, but avoids fractions until the value of the unknown is found, providing, of course, that there are no fractional coefficients in the original equation.

The solution of a quadratic equation by completing the square involves the following steps:

Transpose so that the terms containing the unknown are in the first member and the remaining terms in the second member.

If the coefficient of x^2 is not a perfect square, multiply or divide both members by a number that will make it a perfect square.

Add to each member a number that will make the first member a perfect square.

Find the square root of each member, writing ± to include both roots of the member not containing the unknown.

Solve the resulting linear equations.

EXERCISES

Solve the following equations by completing the square:

1. $2x^2 - 5x = 12$
2. $x^2 - 13x + 22 = 0$
3. $b^2 + 9b = 52$
4. $y^2 - 5y = 66$
5. $d^2 + 5d = -3$
6. $3x^2 - 2x = 1$
7. $7s^2 + 2s = 7$
8. $3v^2 + 10v = 32$
9. $2b^2 - 5b = 3$
10. $6x^2 + 13x = 85$

11. $3y^2 - 5y = 25$

12. $2x^2 - 9x = 4$

13. $4s^2 - 8s = 21$

14. $9x^2 - 26 = 6x$

15. $3x^2 + 5x = 2$

16. $4k^2 - 11k = 3$

17. $12x^2 + 3 = 14x$

18. $2b^2 - 7b + 6 = 0$

19. $6m^2 - m = 5$

20. $4c^2 + c - 5 = 0$

21. $3x^2 + 8 = 5x$

22. $3n^2 - 17n = 28$

23. $10x^2 - 9 = 43x$

24. $2c^2 + 2c = 3$

25. $9x^2 + 6x - 4 = 0$

26. $12m^2 - 13m = 22$

27. $x^2 + \frac{3x}{2} = 17\frac{1}{2}$

28. $3x^2 + 7x = 6$

29. $5u^2 - 8u = 12$

30. $x^2 - 12x + 16 = 0$

31. $2r^2 - 5r = 117$

32. $3c^2 + 7c - 8 = 0$

33. $5h^2 = 3h + 2$

34. $3x^2 + x - 14 = 0$

35. $6d^2 - 5 = -7d$

36. $3x^2 = 42 - 5x$

37. $3z^2 + 50 = 25z$

38. $y + 2 = \frac{3y^2 - 4}{2y + 1}$

39. $4a - \frac{14 - a}{a + 1} = 14$

40. $2ax^2 + 3kx = 2b^2$

Solution:

Multiplying each member by $8a$,

$$16a^2x^2 + 24akx = 16ab^2$$

Completing the square by adding $9k^2$ to each member,

$$16a^2x^2 + 24akx + 9k^2 = 16ab^2 + 9k^2$$

Extracting the square root of each member,

$$4ax + 3k = \pm\sqrt{16ab^2 + 9k^2}$$

Subtracting $3k$ from each member,

$$4ax = -3k \pm \sqrt{16ab^2 + 9k^2}$$

Dividing each member by $4a$, $x = \dfrac{-3k \pm \sqrt{16ab^2 + 9k^2}}{4a}$

41. $x^2 + ax = c$

42. $3x^2 - 2x = k$

43. $5x^2 - bx = m$

44. $ax^2 - 2x = b$

45. $2x^2 + 2ax = 25$

46. $c^2 + 2bc = b^2 - a^2$ (Solve for c)

47. $aw^2 - sw = k$ (Solve for w)

48. $3by^2 + 2by = c$

49. $2\pi R(R + r) = T$ (Solve for R)

50. $s = vt + \frac{1}{2}at^2$ (Solve for t)

135. Solution of Quadratic Equations by Formula. The equation $ax^2 + bx + c = 0$ represents any quadratic equation, since a, b, and c may stand for any numbers. If $a = 5$, $b = 4$, and $c = 8$, the equation becomes $5x^2 + 4x + 8 = 0$. If $a = 3$, $b = 10$, and $c = -25$, the equation becomes $3x^2 + 10x - 25 = 0$.

Since $ax^2 + bx + c = 0$ represents any quadratic equation, the solution of it is the solution of any quadratic equation.

We shall, then, proceed to solve the equation,

$$ax^2 + bx + c = 0$$

Subtracting c from each member, $ax^2 + bx = -c$

Multiplying each member by $4a$, $4a^2x^2 + 4abx = -4ac$

Completing the square by adding b^2 to each member,

$$4a^2x^2 + 4abx + b^2 = b^2 - 4ac$$

Extracting the square root of each member,

$$2ax + b = \pm\sqrt{b^2 - 4ac}$$

Whence, $2ax = -b \pm \sqrt{b^2 - 4ac}$

and $$\mathbf{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

ILLUSTRATIVE EXERCISES

1. Solve $3x^2 - 2x = 8$

Solution:

Writing the equation in the standard form, we have,

$$3x^2 - 2x - 8 = 0$$

Here, $a = 3$, $b = -2$, and $c = -8$. Hence, substituting these values as indicated in the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 3 \cdot (-8)}}{2 \cdot 3}$$

$$= \frac{2 \pm \sqrt{4 + 96}}{6} = \frac{2 \pm 10}{6} = 2 \text{ or } -\frac{4}{3}$$

Checking for $x = 2$, $3 \cdot 4 - 2 \cdot 2 = 8$, or $12 - 4 = 8$.

Check for $x = -\frac{4}{3}$.

2. Solve $3x^2 + 10x - 25 = 0$

In this equation, $a = 3$, $b = 10$, and $c = -25$. Substituting these values in the above formula,

we have, $$x = \frac{-10 \pm \sqrt{100 - 4 \cdot 3 \cdot (-25)}}{6} =$$

$$\frac{-10 \pm \sqrt{100 + 300}}{6} = \frac{-10 \pm 20}{6} = \frac{5}{3} \text{ or } -5$$

Checking for $x = \frac{5}{3}$,

$$3 \cdot \frac{25}{9} + 10 \cdot \frac{5}{3} - 25 = \frac{25}{3} + \frac{50}{3} - 25 = 0$$

Check for $x = -5$.

Solve the following equations by the use of the formula.

1. $x^2 - 3x - 18 = 0$
2. $x^2 + 2x - 5 = 0$
3. $a^2 - 8a + 6 = 0$
4. $3c^2 - 5c - 12 = 0$
5. $4m^2 + 7m = 11$
6. $3s^2 - 4s = 6$
7. $5h^2 = 7h + 8$
8. $6x^2 + 5x - 6 = 0$
9. $x^2 + 9x = 20$
10. $x^2 - 1 = -8x$
11. $6n^2 + 6n = 13$
12. $2p^2 - 8p + 5 = 0$
13. $10x^2 - x = 2$
14. $3b^2 + 14b = 192$
15. $7x^2 = 24x - 19$
16. $6x^2 + 4x = 2x + 20$
17. $2cx^2 - ckx - 2x + k = 0$

Solution:

Writing this in the form $2cx^2 - (ck + 2)x + k = 0$ we have $a = 2c$, $b = -(ck + 2)$, and $c = \text{k}$. Substituting these values in the formula, we obtain,

$$x = \frac{ck + 2 \pm \sqrt{[-(ck + 2)]^2 - 8ck}}{4c}$$

$$= \frac{ck + 2 \pm \sqrt{c^2k^2 + 4ck + 4 - 8ck}}{4c}$$

$$= \frac{ck + 2 \pm (ck - 2)}{4c} = \frac{2ck}{4c} \text{ or } \frac{4}{4c} = \frac{k}{2} \text{ or } \frac{1}{c}.$$

18. $x^2 + mx + n = 0$
19. $3x^2 - kx - c = 0$
20. $2y^2 - 2ay - s = 0$
21. $cy^2 + 3by + m = 0$
22. $2x^2 - (a - 2m)x - am = 0$
23. $x^2 + (a - 3)x = 3a$
24. $sx^2 + (b - 2s)x = 2b$
25. $rx^2 - 2sx + 3t^2 = 0$
26. $x(x - m) = n(n + m)$
27. $x^2 + 2x = cx + 2c$
28. $x^2 - (r + 1)x + r = 0$
29. $x^2 + lx + mx + lm = 0$
30. $bx^2 + cx + d = 0$
31. $(r - s)x^2 - sx = r$
32. $\frac{a}{4} - \frac{21 - a}{4 - a} = 1$

136. An equation of higher degree than the second may sometimes be solved in the same manner that a quadratic equation is solved.

ILLUSTRATIVE EXERCISES

1. Solve $x^4 + 3x^2 = 28$

Subtracting 28 from each member, $x^4 + 3x^2 - 28 = 0$

Factoring, $(x^2 + 7)(x^2 - 4) = 0$

Whence, $x^2 + 7 = 0$, or $x = \pm\sqrt{-7}$

and $x^2 - 4 = 0$, or $x = \pm 2$

The roots of the given equation are 2, -2, $\sqrt{-7}$, and $-\sqrt{-7}$.

Check for $x = 2$: $16 + 12 = 28$.

Check for $x = -2$: $16 + 12 = 28$.

2. Solve $(x^2 - 3x)^2 - 3(x^2 - 3x) = 10$ (1)

Subtracting 10 from each member,

$$(x^2 - 3x)^2 - 3(x^2 - 3x) - 10 = 0 \quad (2)$$

Factoring, $(x^2 - 3x - 5)(x^2 - 3x + 2) = 0$ (3)

Whence, $x^2 - 3x - 5 = 0$ (4)

and $x^2 - 3x + 2 = 0$ (5)

Solving (4) by formula, $x = \dfrac{3 \pm \sqrt{9 + 20}}{2} = \dfrac{3 \pm \sqrt{29}}{2}$

Solving (5) by factoring, $x = 2$ or 1

Check for the last two roots.

Solve the following equations:

1. $x^4 - 5x^2 - 36 = 0$

2. $x^4 + 9x^2 - 36 = 0$

3. $x^4 - 13x^2 = -36$

4. $x^4 + 3x^2 = 40$

5. $9y^4 + 12 = 31y^2$

6. $4z^4 - 5z^2 = -1$

7. $2x^4 - 13x^2 = -20$

8. $4y^4 = 9y^2 - 2$

9. $2x^4 - 13x^2 - 45 = 0$
10. $3x^4 - 67x^2 - 200 = 0$
11. $3x^4 + x^2 - 200 = 0$
12. $(x^2 - x)^2 - 8(x^2 - x) + 12 = 0$
13. $(x^2 - 2x)^2 - 7(x^2 - 2x) - 8 = 0$
14. $(x^2 + 5x)^2 - 8(x^2 + 5x) + 12 = 0$

Sometimes an equation of the third degree may be separated into two equations, one of the first and the other of the second degree.

ILLUSTRATIVE EXERCISE

Solve $x^3 = 1$.

Solution:

Subtracting 1 from each member, $x^3 - 1 = 0$

Factoring, $(x - 1)(x^2 + x + 1) = 0$

Whence, $x - 1 = 0$

and $x = 1$

and $x^2 + x + 1 = 0$

Solving the last equation by formula, $x = \dfrac{-1 \pm \sqrt{-3}}{2}$

The three roots of the equation are therefore 1, $\dfrac{-1 + \sqrt{-3}}{2}$, and $\dfrac{-1 - \sqrt{-3}}{2}$.

Since the cube of each of these three values of x equals 1, they are the three cube roots of 1.

EXERCISES

Solve the following equations:

1. $x^3 = 8$
2. $x^3 = 27$
3. $x^3 + 1 = 0$
4. $x^3 - x^2 - x + 1 = 0$
5. $x^3 + x^2 - 4x - 4 = 0$
6. $x^3 - 3x^2 - 2x + 6 = 0$

137. Imaginary Numbers.

In the last solution shown the number $\sqrt{-3}$ appeared. To find the value of this or any similar number, it would be necessary to find a number which squared would produce -3. Any real number, whether positive or negative, if squared must produce a positive number; therefore such numbers as $\sqrt{-3}$ are called **imaginary.**

An **imaginary** number is the square root of a negative number.

All other numbers are called **real** numbers.

A **complex** number is the sum or difference of a real number and an imaginary number. For example, $2+\sqrt{-3}$ and $2-\sqrt{-3}$ are complex numbers. $a+b\sqrt{-1}$ is the general form of a complex number. $\sqrt{-1}$ is usually denoted by the letter i. The general form of the complex number is, then, $a+bi$.

Just as irrational numbers and negative numbers were at first considered meaningless and were later given a meaning, so imaginary numbers have been given a meaning. These numbers will be considered more fully in a later course.

MISCELLANEOUS EXERCISES

138. Solve the following equations by the method that seems best in each case. If the factors can be readily found, the factoring method offers the quickest solution. Otherwise the solution by formula is preferable.

1. $x^2-3x=10$
2. $x^2+5x=14$
3. $m^2-3m-28=0$
4. $r^2-8r=0$
5. $x^2-x-72=0$
6. $2a^2+a-28=0$
7. $s^2-3s+5=0$
8. $n^2-3=0$

9. $4y^2 - 7y + 2 = 0$

10. $3c^2 - 13c = -10$

11. $5x^2 + 3x - 12 = 0$

12. $4b^2 + 3b - 45 = 0$

13. $(x + 1)(x - 4) = 50$

14. $\dfrac{2}{x - 1} - 10x = 9$

15. $6 = \dfrac{x}{2x^2 - 1}$

16. $x + \dfrac{1}{2} = 2 + \dfrac{1}{x}$

17. $m + \dfrac{1}{5} = 5 + \dfrac{1}{m}$

18. $\dfrac{x}{x - 3} + \dfrac{8}{x + 2} = 3$

19. $\dfrac{3}{x - 5} - 5 = \dfrac{2x}{x - 3}$

20. $\dfrac{1}{m} - \dfrac{1}{2} = \dfrac{1}{m - 2}$

21. $\dfrac{7c}{8} + \dfrac{8}{7c} = \dfrac{9}{10}$

22. $\dfrac{5p}{7} + \dfrac{1}{p} = \dfrac{7}{5}$

23. $2x^2 + .3x - 51.5 = 0$

24. $\dfrac{3}{x} + x = \dfrac{7}{2}$

25. $\dfrac{x^2}{5} - \dfrac{2x}{3} = \dfrac{2}{15}$

26. $x^2 - ax + b = 0$

27. $\dfrac{1}{x} + x = 3 + \dfrac{3}{x}$

28. $\dfrac{15x}{x - 1} = 11(x - 1)$

29. $\dfrac{1}{r} - \dfrac{1}{6} = \dfrac{1}{r - 6}$

30. $\dfrac{m^2}{x^2} = \dfrac{m + 1}{x + 1}$

31. $\dfrac{r}{y - s} + \dfrac{s}{y - r} = 2$

32. $x^2 + b^2 = m^2 + 2bx$

33. $\dfrac{x - 1}{2} - \dfrac{3x + 1}{3} = -\dfrac{4x^2}{3}$

34. $\dfrac{x + 1}{x - 1} + \dfrac{3x + 5}{x + 1} = \dfrac{29}{6}$

35. $\dfrac{x}{x + 1} + \dfrac{x + 1}{x} = \dfrac{113}{56}$

36. $\dfrac{1}{x + 1} - \dfrac{1}{x + 2} = \dfrac{1}{x + 3}$

37. $(c+d)x^2 - (2c+d)x + c = 0$

38. $x^2 + 4ab = 2x(b + a)$

39. $\dfrac{x}{a^2} + \dfrac{a^2}{x} = \dfrac{x}{b^2} + \dfrac{b^2}{x}$

40. $\dfrac{x - 4c + d}{d - c} = \dfrac{9d + x}{x}$

41. $x(2x-a)+x(x-a)=bx$

42. $(a-x)^2+(b-x)^2=(b-a)^2$

43. $(x+b)^2+6(x+b)+9=0$

139. Problems Solved by Quadratic Equations.

1. The sum of two numbers is 16 and their product is 48. What are the numbers?

Solution:

Let $x =$ one number,

then $16-x =$ the other,

and by the conditions of the problem,

$$x(16-x)=48$$

Solving this equation, we find $x = 4$ or 12.

If $x=4$, $16-x=12$.

If $x=12$, $16-x=4$.

Hence, the two numbers are 4 and 12. Since the sum of 12 and 4 is 16 and their product is 48, these two numbers satisfy the conditions of the problem and the results are correct.

2. The difference between two numbers is 9 and their product is 36. Find the numbers.

3. The square of a number is 36 greater than 5 times the number. What is the number?

4. Find the dimensions of a field which is 20 rods longer than it is wide, and contains 15 acres.

5. The distance around a lot is 120 yards and its area is 800 square yards. Find its dimensions.

6. Two numbers differ by 4. The smaller exceeds the quotient of the larger divided by the smaller by 2. Find the numbers.

Let $x =$ the larger number.
then $x - 4 =$ the smaller,
and $$x - 4 - \frac{x}{x-4} = 2.$$

Solving this equation, we find $x = 3$ or 8. Hence, the larger number is either 3 or 8. If the larger number is 3, the smaller is -1. If the larger number is 8, the smaller is 4. Show that both results satisfy the conditions of the problem.

7. The pages of a book are $2\frac{1}{2}$ inches longer than they are wide and each page contains $31\frac{1}{2}$ square inches. What are the dimensions of a page?

8. A picture 8 inches by 10 inches is placed in a frame of uniform width. The area of the frame is half that of the picture. How wide is the frame?

9. A rug placed on a floor whose length is 4 feet greater than its width leaves a margin of 2 feet all around the rug. The area of this margin is equal to the area of the rug. What are the dimensions of the floor?

10. Find a number whose square exceeds four times the number by 5.

11. Find two consecutive integers whose product is 306.

12. Find three consecutive integers such that 4 times the largest is 2 less than the product of the other two.

13. One leg of a right triangle is 7 inches longer than the other and the hypotenuse is 13 inches long. Find the length of each leg.

14. The hypotenuse of a right triangle is 4 inches longer than one of its legs. The second leg is 8 inches. Find the length of the first leg and of the hypotenuse.

15. A number is 4 times its reciprocal. What is the number?

16. The denominator of a fraction is 2 greater than the numerator. The reciprocal of the fraction is $\frac{16}{15}$ greater than the original fraction. What is the fraction?

Hint. Let x be the numerator.

17. The numerator of a fraction exceeds the denominator by one. If 2 be added to each term of the fraction, the value of the fraction is diminished by $\frac{1}{4}$. What is the fraction?

18. A merchant paid $1.80 for some eggs. Later, when eggs were 3 cents a dozen cheaper, he bought 2 dozen more for the same amount. What was the price per dozen and how many dozen did he buy in each transaction?

19. If the price of oranges were 2 cents apiece less, 8 more could be bought for 60 cents. What is the price?

20. A fruit merchant bought some lemons for $5. He sold all except 2 dozen at an advance of 10 cents a dozen and gained $1.90. How many did he buy and what was the cost per dozen?

21. A man ran 220 yards in a certain time. If he had run 1 yard per second faster, his time would have been 2 seconds less. What was his time?

22. A and B ran a race of 100 yards. A's rate was 1 foot per second faster than B's and A won by half a second. What were the rate and time of each?

23. In a 220-yard race A won from B by half a second. A's rate was $\frac{2}{9}$ of a yard per second faster than B's. What were the rate and time of each?

24. A can run 100 yards in 2 seconds less time than B. A's rate is 5 feet per second faster than B's. What are the time and rate of each?

25. A passenger train runs 150 miles in $2\frac{1}{2}$ hours less time than a freight train. The latter runs 10 miles an hour slower than the former. What is the rate of each?

26. A can do a piece of work in 2 days less time than B can. Both can do $\frac{7}{24}$ of the work in one day. In how many days can each do the work alone?

Solution:

Let x = the number of days it takes A to do the work.
Then $x+2$ = the number of days it takes B to do the work.

What part of the work can A do in one day? B? What expression represents the part of the work both can do in one day? What does this expression equal?

27. A can do a piece of work in 4 days less than B and both can do $\frac{3}{8}$ of the work in one day. In how many days can each do the work alone?

28. It takes A 3 days longer to finish a job than B. Both can do the work in $3\frac{3}{5}$ days. In how many days can each do the work alone?

Hint. Both can do $\frac{1}{3\frac{3}{5}} = \frac{5}{18}$ of the work in one day.

29. A train on the Pennsylvania R. R. running from New York to Chicago (900 miles) was 2 hours late on its arrival at Chicago. The rate was 5 miles an hour less than the schedule. What was the rate?

30. A passenger train on the New York Central line leaves New York for Buffalo (440 miles) at the same time that a freight train leaves Albany for Buffalo (300 miles). The passenger train runs 25 miles an hour faster than the freight and arrives in Buffalo 2 hours before the freight. What is the rate of each?

31. A boatman can row 21 miles down a stream flowing 2 miles an hour in 4 hours less time than he can row the same distance up the stream. How fast can he row in still water?

32. A boatman can row 6 miles an hour in still water. It takes him 3 hours longer to row 24 miles up a stream than it does to row the same distance down the stream. How fast does the stream flow?

33. The circumference of the hind wheel of a wagon is 3 feet greater than that of the fore wheel. In going 120 yards the fore wheel makes 10 more revolutions than the hind wheel. What is the circumference of each?

Hint. If x is the circumference of the hind wheel, how far does the wagon go while the hind wheel makes one revolution? How many revolutions will the hind wheel make in going 120 yards?

34. The circumference of the drive wheel of a railroad engine is 10 feet greater than that of a car wheel. In going a mile the car wheel makes 550 more revolutions than the drive wheel. What is the circumference of each?

35. An estate worth $15,000 was divided equally among a certain number of persons. If there had been 2 more persons, each would have received $2000 less. How many persons were there and how much did each receive?

36. A man's wages for a certain time were $24. If he had received half a dollar less per day, he would have had to work 4 days longer to earn the same amount. What were his daily wages?

37. A travels east at 4 miles an hour and B north at 3 miles an hour. In how many hours will they be 30 miles apart?

Hint. A and B are traveling along the legs of a right triangle, and the distance between them is the hypotenuse.

38. A travels east and B south. A goes 2 miles an hour faster than B. At the end of 2 hours, they are 20 miles apart. How fast does each travel?

REVIEW EXERCISES

140. **1.** What is the degree of $x^2y-2xy=3xy-4y^2$ with respect to x? With respect to y? With respect to x and y?

2. State the degree of each of the following equations:

(a) $x^2 - \frac{3}{x} = 2x - 1$

(b) $x + 3 - \frac{2}{x} = 3$

3. What is a quadratic equation? What is the general quadratic equation?

4. Solve by factoring:

(a) $x^2 + 17x - 38 = 0$
(b) $x^2 - 15x = -26$
(c) $d^2 - 3d = 154$
(d) $y^4 - 13y^2 + 36 = 0$

5. Solve by completing the square:

(a) $x^2 - 5x = 14$
(b) $12y^2 = 15y + 20$
(c) $4x^2 - 12x = 6$
(d) $\frac{6}{x} - \frac{1}{6} = \frac{2x}{x+1}$

6. Solve by formula:

(a) $3x^2 - 8x + 2 = 0$
(b) $10x + 3x^2 = 12x - 4$
(c) $\frac{x^2}{3} - 4x = 3$
(d) $\frac{x^2-3}{x+1} - \frac{x^2+1}{x-1} = 3$

7. The formula $\frac{W}{P} = \frac{D^2}{d^2}$ is used in physics. Solve for D and d.

8. In the formula of problem 7, find d if $W = 100$, $P = 25$, and $D = 4$.

9. The area of a certain rectangle is 3 times that of a square. The length of the rectangle is 4 feet greater than the width of the rectangle and twice the length of a side of the square. Find the dimensions of the rectangle and of the square.

SUMMARY

SECTION

132. Definition of quadratic equation. General quadratic equation, $ax^2 + bx + c = 0$.
133. Solution of quadratic equations by factoring.
134. Solution of quadratic equations by completing the square.
135. Solution of quadratic equations by formula.
136. Solution of equations of higher degree than the second.
137. Imaginary numbers. Complex numbers.
138. Miscellaneous exercises.
139. Problems solved by quadratic equations.
140. Review exercises.

Did John find Lesta's
Rubbers,
I didnd know girls
had to wear rubbers
You sure are green

CHAPTER XXI

SIMULTANEOUS QUADRATICS

141. Linear-Quadratic System.

Not all systems of simultaneous quadratic equations in two unknowns can be solved by the methods already given.

For example, take the system of equations:

$$x^2 + y = 11 \qquad (1)$$

$$x + y^2 = 7 \qquad (2)$$

Solving (1) for y, $$y = 11 - x^2 \qquad (3)$$

Substituting the value found for y in (2),

$$x + 121 - 22x^2 + x^4 = 7$$

or $$x^4 - 22x^2 + x = -114$$

This is an equation of the fourth degree and our present knowledge does not enable us to solve it.

The most important of the types of simultaneous quadratic equations that can be solved without involving the solution of an equation of higher degree than the second, is the linear-quadratic type; that is, systems of two equations, one of which is linear and one quadratic. The scope of this book limits our treatment of simultaneous quadratic equations to this type.

ILLUSTRATIVE EXERCISE

Solve the equations:

$$x^2 + 3xy = 55 \qquad (1)$$

$$2x - y = 8 \qquad (2)$$

Solution:

Solving (2) for y, $$y = 2x - 8 \qquad (3)$$

Substituting in (1) the value of y given in (3),

$$x^2 + 3x(2x - 8) = 55 \qquad (4)$$

Whence, $$7x^2 - 24x - 55 = 0 \qquad (5)$$

Solving (5), $x = 5$ or $-\frac{11}{7}$

Substituting 5 for x in (1), $y = 10 - 8 = 2$

Substituting $-\frac{11}{7}$ for x in (3),

$$y = -\frac{22}{7} - 8 = -\frac{78}{7}$$

There are, therefore, two solutions, *viz.*:

$$x = 5 \qquad x = -\frac{11}{7}$$
$$y = 2 \qquad y = -\frac{78}{7}$$

Check for $x = -\frac{11}{7}$, $y = -\frac{78}{7}$.

Substituting these values for x and y in (1) and (2),

$$\left(-\frac{11}{7}\right)^2 + 3\left(-\frac{11}{7}\right)\left(-\frac{78}{7}\right) = \frac{121}{49} + \frac{2574}{49}$$
$$= \frac{2695}{49} = 55$$

and $$2\left(-\frac{11}{7}\right) - \left(-\frac{78}{7}\right) = -\frac{22}{7} + \frac{78}{7} = \frac{56}{7} = 8$$

Equation (5) might have been solved by factoring, *viz.*:

$$(x - 5)(7x + 11) = 0$$

Whence, $x = 5$ or $-\frac{11}{7}$

All systems of equations of this type can be solved in the manner indicated above. Equation (2) might have been solved for x in terms of y, and the value thus obtained substituted for x in (1). But this would have necessitated more substituting than the method used and also would have involved fractions. Before beginning the solution, the student should observe for which of the unknowns it is better to solve the linear equation. Usually it is preferable to select the one which will give the less work in substituting.

The steps may be stated as follows:

I. Solve the linear equation for one of the unknowns in terms of the other.

II. Substitute the value thus obtained for the unknown in the quadratic equation.

III. Solve the resulting quadratic equation.

IV. Substitute the values obtained in step III in the equation obtained in step I. This will give the values of the other unknown.

EXERCISES

Solve the following systems of equations:

1. $x + y = 8$
$xy = 12$

2. $2x - y^2 = 8$
$x - y = 4$

3. $x^2 + y^2 = 20$
$x + 2y = 8$

4. $xy + 2y^2 = 39$
$x + y = 10$

5. $2x + y = 14$
$x^2 + 2xy = 44$

6. $x - 3y = 2$
$xy - x = 8$

7. $2x - 5y = 5$
$3xy + 5y^2 = 2$

8. $x^2 + y^2 = 45$
$x + y = 9$

9. $y = 3x$
$y^2 - xy = \frac{10}{3}$

10. $r - 2s = 5$
$2rs - r = 7$

11. $m + 3n = 15$
$2m^2 - n^2 = 2$

12. $x^2 + xy + y^2 = 19$
$3x - 4y = 1$

13. $5xy - 3y^2 = 2x$
$3x + y = 12$

14. $6m^2 - k^2 = 29$
$2m - k = 1$

15. $4r^2 - 3z^2 = -3$
$2r + \frac{z}{2} = 4$

16. $3c + 4d^2 = 5.5$
$2c + 4d = 5$

17. $4xy - y^2 = 35.2$
$x = .8y$

18. $xy + 16x^2 = 10$
$y - x = \frac{7}{12}$

19. The sum of two numbers is 17 and the sum of their squares is 169. What are the numbers?

20. The difference between two numbers is 3 and their product is 180. What are the numbers?

21. The perimeter of a rectangle is 36 feet and its area is 80 square feet. What are its dimensions?

22. The hypotenuse of a right triangle is 17 inches and the sum of its other two sides is 23 inches. What are the lengths of the sides?

23. The sum of the squares of two consecutive even numbers is 340. What are the numbers?

24. An automobile makes a trip of 30 miles and returns, making the entire distance in $4\frac{1}{2}$ hours. The rate returning was 3 miles an hour faster than that going. What was the rate each way?

142. Graphs of Quadratic Equations in Two Unknowns.

The graph of a quadratic equation in two unknowns may be plotted in the same manner as the graph of a linear equation in two unknowns. That is, several solutions of the equation are found and the corresponding points plotted. A smooth curve drawn through these points is the graph of the equation.

ILLUSTRATIVE EXERCISE

Plot the graph of $y^2 = 4x + 4$.

Solving the equation for y,

$$y = \pm 2\sqrt{x+1}$$

We now find several solutions of the equation by assigning values to x and determining the corresponding values of y. Thus, if $x = 0$, $y = \pm 2$; if $x = 1$, $y = \pm\sqrt{8} = \pm 2.8$, etc. The solutions may be conveniently arranged as in the accompanying table.

x	y
0	± 2
1	± 2.82
2	± 3.46
3	± 4
-1	0
-2	$\pm\sqrt{-8}$ (Imag.)

If x is less than -1, y is the square root of a negative number, and is, therefore, imaginary. We can not then give x any value less than -1.

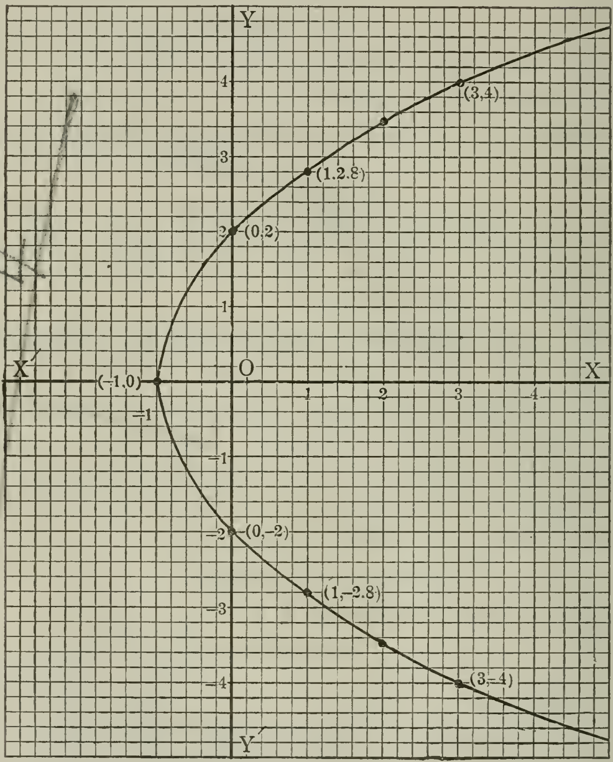


Figure 1. Graph of $y^2 = 4x + 4$

Plotting the points (0, 2), (0, − 2), (1, 2.82), (1, − 2.82), etc., and drawing a smooth curve through them we have the graph shown in figure 1. This curve is called a **parabola.**

EXERCISES

Plot the graph of each of the following:

1. $y^2 = 2x$ **2.** $y^2 = 3x - 6$ **3.** $x^2 = 4y$ **4.** $x^2 = 2y + 3$

ILLUSTRATIVE EXERCISE

Plot the graph of $4x^2 + 9y^2 = 36$

Solving for y,

$$y = \pm \tfrac{2}{3}\sqrt{9 - x^2}$$

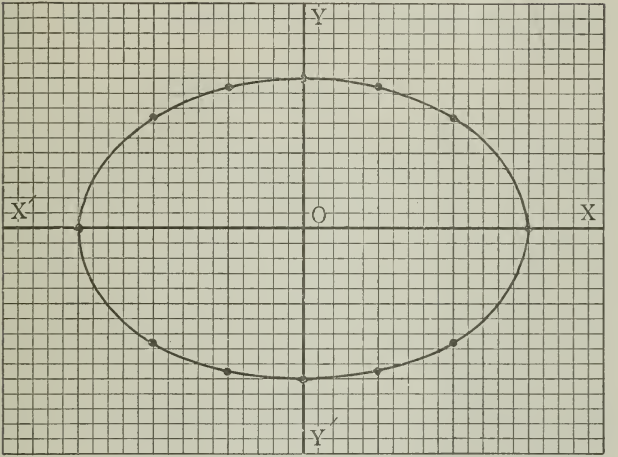


Figure 2. Graph of $4x^2 + 9y^2 = 36$

Proceeding as in the previous problems, we obtain the accompanying table of solutions:

x	y
0	± 2
± 1	± 1.88
± 2	± 1.48
± 3	± 0
± 4	(Imag.)

If x is numerically greater than 3, y is imaginary. Plotting the points corresponding to the above solutions, and drawing a smooth curve through them, the graph in figure 2 is obtained. The graph is an **ellipse.**

The graph of an equation of the form $x^2 + y^2 = 16$ is a circle, which is a special form of the ellipse.

EXERCISES

Plot the graph of each of the following equations:

1. $x^2 + 4y^2 = 4$
2. $x^2 + y^2 = 16$
3. $x^2 + y^2 = 9$
4. $4x^2 + 25y^2 = 100$
5. $x^2 + y^2 = 25$
6. $9x^2 + 16y^2 = 144$

ILLUSTRATIVE EXERCISE

Plot the graph of $3x^2 - 4y^2 = 12$

Solving this equation for y,

$$y = \pm \tfrac{1}{2}\sqrt{3x^2 - 12}$$

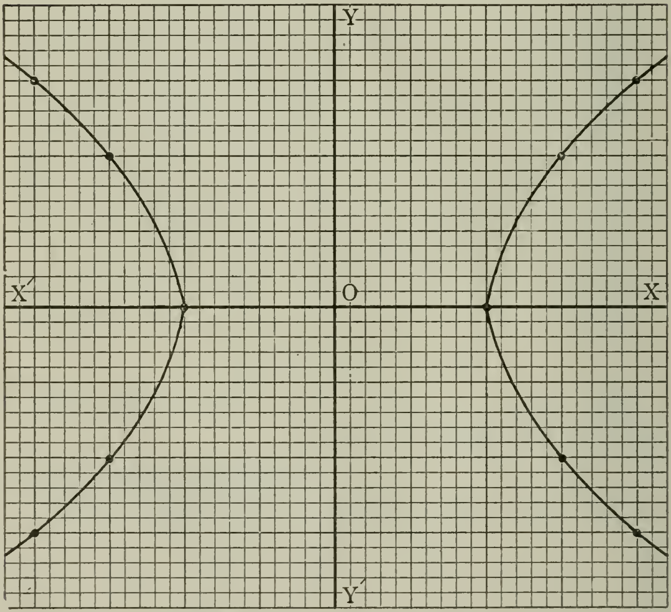


Figure 3. Graph of $3x^2 - 4y^2 = 12$

The following table gives a number of solutions: When x is numerically less than 2, y is imaginary.

x	y
± 2	0
± 3	± 1.94
± 4	± 3

Plotting the points corresponding to these solutions, the graph in figure 3 is obtained. It is an **hyperbola.**

EXERCISES

Plot the graph of each of the following equations:

1. $4x^2 - y^2 = 4$
2. $4x^2 - 9y^2 = 36$
3. $9x^2 - 16y^2 = 144$
4. $4x^2 - 25y^2 = 100$

143. Solution of Simultaneous Quadratic Equations by Graphs.

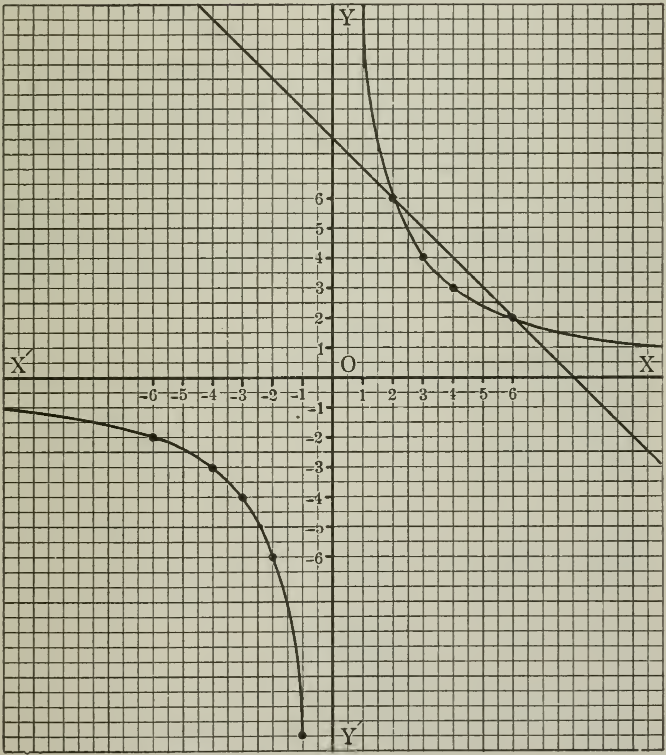


Figure 4. Graph of $x + y = 8$ and $xy = 12$

Solve graphically
$$x + y = 8 \tag{1}$$
$$xy = 12 \tag{2}$$

Plot the graphs of (1) and (2) on the same axes. Since the co-ordinates of any point on the graph of (1) are a solution of (1) and the co-ordinates of any point on the graph of (2) are a solution of (2), the co-ordinates of an intersection of the two graphs are a solution of both (1) and (2). The two graphs, as shown in figure 4, intersect in the two points (2, 6) and (6, 2). Hence, the two solutions of (1) and (2) are $x = 6$, $y = 2$, and $x = 2$, $y = 6$.

Check by solving the system in the usual manner.

The solution by graphs may be checked by substituting the values found for the unknowns in the given equations. It is necessary to bear in mind, however, that, owing to inaccuracy in plotting the graphs and inability to estimate the co-ordinates of the intersection accurately, the solutions may only approximately satisfy the equations.

EXERCISES

Solve the following systems graphically:

1. $2x + y = 6$
   $x^2 + y^2 = 9$
2. $x + y = 5$
   $y^2 = 2x$
3. $x^2 + 4y^2 = 4$
   $3x - y = 3$
4. $x^2 + y^2 = 16$
   $4x^2 + 9y^2 = 36$
5. $xy = 10$
   $x^2 + y^2 = 25$
6. $x^2 - 4y^2 = 4$
   $x - 2y = 1$

SUMMARY

SECTION

141. Linear-quadratic systems of equations.
142. Graphs of quadratic equations in two unknowns.
143. Solution of simultaneous quadratic equations by graphs.

CHAPTER XXII

EXPONENTS AND RADICALS

144. Review of Positive Integral Exponents.

A positive integral exponent has been defined. (See section 7.) The following laws for positive integral exponents have been used:

I. $a^m \cdot a^n = a^{m+n}$ Index law of multiplication.

II. $a^m \div a^n = a^{m-n}$ Index law of division.

III. $(a^m)^n = a^{mn}$ Index law of involution.

IV. $\sqrt[n]{a^m} = a^{\frac{m}{n}}$ Index law of evolution.

V. $(ab)^n = a^n b^n$ $\quad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

VI. $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ $\quad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

EXERCISES

Perform the following indicated operations:

1. $x^2 \cdot x^3$
2. $x^3 \cdot x^n$
3. $a^n \cdot a$
4. $x^7 \div x^2$
5. $a^m \div a^3$
6. $\dfrac{a^{m+2}}{a^m}$
7. $(y^3)^5$
8. $(k^3)^2$
9. $(s^m)^2$
10. $\sqrt{a^{10}}$
11. $\sqrt[3]{x^{15}}$
12. $\sqrt[4]{m^{12}}$
13. $\sqrt[3]{x^9}$
14. $(a^2 b^3)^4$
15. $(2^3 \cdot 5^2)^3$
16. $(c^n b^3)^2$
17. $\sqrt[3]{27a^6}$
18. $\sqrt[5]{2^5 x^{10}}$
19. $\sqrt[3]{a^9 b^{12}}$
20. $(2^2 x^3)^5$
21. $a^n \cdot a^{n+2}$
22. $\left(\dfrac{2a^2}{b^2}\right)^4$
23. $\sqrt{\dfrac{4x^2}{9y^4}}$
24. $\sqrt[3]{\dfrac{8a^3}{27c^6}}$
25. $(2^2 a^2 c^3)^4$
26. $\left(\dfrac{3a}{2b^2}\right)^3$

145. Extension of the Index Laws.

So far, formula II in the preceding section has been used only when m was greater than n. If this restriction (m greater than n) be removed, we may have the zero exponent ($m = n$), or a negative exponent (m less than n). For example,

$$a^3 \div a^3 = a^{3-3} = a^0, \text{ and } a^2 \div a^5 = a^{2-5} = a^{-3}.$$

Similarly, law IV in the preceding section, has thus far been applied only when n is a factor of m. If this restriction be removed, we shall have a fractional exponent, *e. g.*, $\sqrt[5]{a^3} = a^{\frac{3}{5}}$

Zero exponent, negative exponent, and fractional exponent may be defined by extending laws II and IV.

146. Zero Exponent.

By extending law II, $\dfrac{a^3}{a^3} = a^{3-3} = a^0$

It is evident that any number divided by itself equals 1.

$$\frac{a^3}{a^3} = 1. \quad \text{Hence, by Axiom I, } a^0 = 1.$$

Any number with the exponent zero equals one.

What is the value of 2^0? 10^0? 97^0? x^0?

147. Negative Exponent.

$$\text{By law II, } \frac{a^2}{a^5} = a^{2-5} = a^{-3}$$

But $\dfrac{a^2}{a^5} = \dfrac{1}{a^3}$ (dividing numerator and denominator by a^2)

Hence, by Axiom I, $a^{-3} = \dfrac{1}{a^3}$.

A number with a negative exponent is equal to the reciprocal of the number found by changing the sign of the exponent.

ILLUSTRATIVE EXERCISES

1. $3^{-2} = \frac{1}{3^2} = \frac{1}{9}.$

2. Change $\frac{2a^{-2}x^3}{3y^{-3}z}$ to an equivalent expression having only positive exponents.

Solution:

$$\frac{2a^{-2}x^3}{3y^{-3}z} = \frac{2 \cdot \frac{1}{a^2} \cdot x^3}{3 \cdot \frac{1}{y^3} \cdot z} = \frac{2y^3x^3}{3a^2z}$$

As shown in the above examples, it follows from the definition of a negative exponent that *a factor may be removed from the numerator to the denominator or from the denominator to the numerator by changing the sign of its exponent.*

EXERCISES

Change the following to equivalent expressions with only positive exponents, and simplify when possible:

1. 3^{-3}

2. 7^{-2}

3. 2^{-5}

4. 6^{-3}

5. 2^{-7}

6. x^{-3}

7. $2b^{-2}$

8. c^{-1}

9. $x^{-2}y^3$

10. $\frac{1}{x^{-2}}$

11. $\frac{1}{3^{-2}}$

12. $\frac{1}{2^{-3}}$

13. $\frac{1}{4^{-3}}$

14. $\frac{2x^{-2}}{2y^{-2}}$

15. $\frac{4(xyz)^{-1}}{3a^{-3}}$

16. $\frac{3^{-2}x^2}{y^{-2}z^3}$

17. $\frac{5x^{-1}y^2z^2}{3^{-1}a^{-3}b^2}$

18. $\frac{cy^3z^{-3}}{zyx^{-3}}$

19. $\frac{2^{-1}x^2z}{3s^{-2}}$

20. $\frac{4m^{-2}b^3}{2^{-2}ac^{-3}}$

21. $4m^0x^{-2}c^4$

22. $\frac{6^{-2}x^{-3}}{25^0m^{-3}}$

23. $5^{-2}m^{-3}(a+2b)^0$

24. $(4c^{-2}b^3)^{-1}(x+3y)^{-1}$

148. Fractional Exponent.

Extending law IV to include all values of m and n, we have

$$\sqrt[n]{a^m} = a^{\frac{m}{n}}$$

or $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

A fractional exponent is interpreted as follows:

The numerator of a fractional exponent indicates a power and the denominator a root.

We shall assume that

$$\sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

ILLUSTRATIVE EXERCISES

$$9^{\frac{3}{2}} = \sqrt{9^3} = \sqrt{3^6} = 3^3 = 27, \text{ or}$$

$$9^{\frac{3}{2}} = (\sqrt{9})^3 = 3^3 = 27$$

EXERCISES

Find the value of each of the following:

1. $16^{\frac{3}{4}}$

2. $27^{\frac{2}{3}}$

3. $81^{\frac{1}{2}}$

4. $100^{\frac{3}{2}}$

5. $125^{\frac{2}{3}}$

6. $8^{\frac{4}{3}}$

7. $64^{\frac{2}{3}}$

8. $49^{\frac{3}{2}}$

9. $81^{\frac{3}{4}}$

10. $4^{-\frac{1}{2}}$

11. $27^{-\frac{1}{3}} = \left(\frac{1}{27}\right)^{\frac{1}{3}} = \frac{1}{3}$

12. $32^{-\frac{2}{5}}$

13. $16^{-\frac{3}{2}}$

14. $\left(\frac{1}{9}\right)^{\frac{1}{2}}$

15. $8^{-\frac{1}{3}}$

16. $25^{-\frac{3}{2}}$

17. $16^{-\frac{3}{4}}$

18. $36^{-\frac{1}{2}}$

19. $64^{-\frac{2}{3}}$

20. $(-64)^{-\frac{1}{3}}$

21. $\left(\frac{4}{9x^4}\right)^{-\frac{3}{2}}$

22. $(-27)^{\frac{2}{3}}$

23. $(\frac{1}{4})^{\frac{3}{2}}$

24. $(\frac{4}{9})^{-\frac{3}{2}}$

25. $(\frac{8}{27})^{-\frac{2}{3}}$

26. $.25^{\frac{3}{2}}$

The definitions of zero, negative, and fractional exponents might have been obtained by extending Law I, making it apply not only when m and n are positive integers, but also when one or both of them are zero, negative numbers, or fractions.

It can be proved that Laws I to VI hold for zero, negative, or fractional exponents as defined above. In this chapter we shall assume them to hold for all exponents, without giving the formal proof.

EXERCISES

149. **1.** State in words what Law I says of the exponent of a number in a product.

Apply Law I to the following products:

2. $a^2 \cdot a^3$
3. $a^2 \cdot a^{-1}$
4. $a^3 \cdot a^0$
5. $x^2 \cdot x^{-3}$
6. $s^{-2} \cdot s^{-3}$
7. $c \cdot c^{-4}$
8. $2^2 \cdot 2^{-3}$
9. $3^4 \cdot 3^{-2}$
10. $5^{-1} \cdot 5^{-2}$
11. $s^{\frac{1}{2}} \cdot s^2$
12. $k^{\frac{1}{3}} \cdot k^{-1}$
13. $h^{\frac{1}{2}} \cdot h^{\frac{1}{3}}$
14. $x^4 \cdot x^{-3}$
15. $s^6 \cdot s^{-5}$
16. $y \cdot y^3 \cdot y^{-2}$
17. $a^{\frac{1}{2}} \cdot a^{-\frac{1}{3}}$
18. $c^2 \cdot c^{-5} \cdot c^{\frac{1}{2}}$
19. $m^{\frac{1}{2}} \cdot m \cdot m^{\frac{1}{4}}$
20. $x^{\frac{3}{4}} \cdot x^{\frac{1}{2}} \cdot x^{-\frac{1}{4}}$
21. $a^{\frac{2}{3}} \cdot a^{\frac{5}{6}} \cdot a^{\frac{1}{2}}$
22. $b^{\frac{3}{4}} \cdot b^{-\frac{2}{3}}$
23. $k^{\frac{1}{2}} \cdot k^{\frac{1}{6}} \cdot k^{-\frac{2}{3}}$
24. $x^2 \cdot x^{-1} \cdot x^0$
25. $v^{\frac{2}{5}} \cdot v^0 \cdot v^{-\frac{3}{10}}$
26. $x^{n+2} \cdot x^{-n}$
27. $x^{m+2} \cdot x^{m-3}$
28. $a^{c+1} \cdot a^{2c}$

EXERCISES

150. **1.** State in words what Law II says of the exponent of a number in a quotient.

Apply Law II to the following quotients:

2. $a^5 \div a^3$
3. $x^3 \div x^5$
4. $y^3 \div y^{-2}$
5. $c^{-1} \div c^{-3}$
6. $2^3 \div 2^5$
7. $3^2 \div 3^{-1}$
8. $5^3 \div 5^4$
9. $a^{-3} \div a^2$
10. $m^{\frac{3}{2}} \div m$

11. $x^{\frac{1}{2}} \div x^{\frac{1}{3}}$

12. $2^{\frac{3}{2}} \div 2^{\frac{5}{6}}$

13. $k^{-3} \div k^{-2}$

14. $x^{\frac{3}{4}} \div x^{-\frac{1}{4}}$

15. $x^{-3} \div x^{-5}$

16. $a^{-2}x^{3} \div ax^{-2}$

17. $m^{2}n^{-1} \div m^{-1}n^{-2}$

18. $3x^{-2} \div x^{3}$

19. $4x^{2}y \div 2xy^{-2}$

20. $9x^{3}z \div 3x^{-1}z^{-\frac{1}{2}}$

21. $6x^{\frac{2}{3}}k^{-\frac{1}{6}} \div 2x^{\frac{1}{3}}k^{-\frac{1}{3}}$

EXERCISES

151. **1.** State in words what Law III says of the exponent of a number in a power of a power.

Apply Law III to the following:

2. $(a^{-3})^{2}$

3. $(x^{-2})^{3}$

4. $(s^{2})^{-2}$

5. $(m^{4})^{\frac{1}{2}}$

6. $(x^{2})^{-4}$

7. $(6^{3})^{\frac{2}{3}}$

8. $(x^{-2})^{-2}$

9. $(z^{\frac{1}{3}})^{6}$

10. $(a^{-\frac{1}{3}})^{-3}$

11. $(z^{\frac{3}{4}})^{-4}$

12. $(c^{-3})^{-\frac{1}{4}}$

13. $(s^{-\frac{5}{3}})^{\frac{3}{2}}$

14. $(m^{-\frac{1}{2}})^{-4}$

15. $(k^{n})^{2}$

16. $(x^{3})^{a}$

EXERCISES

152. **1.** State in words what Law IV says of the exponent of a number in a root of a power.

Apply Law IV to each of the following:

2. $\sqrt{2^{10}}$

3. $\sqrt{2^{5}}$

4. $\sqrt{m^{6}}$

5. $\sqrt{8} =$
$\sqrt{2^{3}} = 2^{\frac{3}{2}}$

6. $\sqrt[3]{x^{9}}$

7. $\sqrt{5^{7}}$

8. $\sqrt[4]{27}$

9. $\sqrt[3]{c^{5}}$

10. $\sqrt{m^{-3}}$

11. $\sqrt[3]{k^{-2}}$

12. $\sqrt{p^{m}}$

13. $\sqrt{x^{-r}}$

EXERCISES

153. **1.** State in words what Law V says about finding a power of a product. According to Law III, what must be done with the exponent of each factor?

2. State in words what Law V says about finding a power of a quotient (fraction). According to Law III, what must be done with the exponent of the numerator and with the exponent of the denominator?

Apply Law V to each of the following:

3. $(a^2b^3)^2$

4. $(mn^2)^3$

5. $(2 \cdot 3)^3$

6. $(4ab)^{-2}$

7. $(3xyz)^{-3}$

8. $(5mnp)^{-4}$

9. $(\frac{2}{3})^3$

10. $\left(\frac{a}{b}\right)^{-3}$

11. $\left(\frac{x}{z}\right)^{\frac{1}{2}}$

12. $\left(\frac{2mn}{x}\right)^{\frac{2}{3}}$

13. $(3 \cdot 5)^{-4}$

14. $(2^{\frac{1}{2}} \cdot 3^{\frac{2}{3}})^{-3}$

15. $(6a^2b^{-2})^{-3}$

16. $(2^{-2} \cdot 7^{-1})^{\frac{1}{2}}$

17. $(5a^2bc^{-2})^{-3}$

18. $\left(\frac{9xy^2}{16m^{-3}}\right)^{-4}$

EXERCISES

154. **1.** State in words what Law VI says about finding a root of a product. According to Law IV, what must be done with the exponent of each factor?

2. State in words what Law VI says about finding a root of a quotient (fraction). According to Law IV, what must be done with the exponent of the numerator and the denominator?

Apply Law VI to each of the following:

3. $\sqrt{4 \cdot 9}$

4. $\sqrt{16 \cdot 9}$

5. $\sqrt{a^2b^4c^6}$

6. $\sqrt[3]{2^3 \cdot 3^6}$

7. $\sqrt{a^3b^5}$

8. $\sqrt{m^3n^2}$

9. $\sqrt[3]{2^{-3}x^{-6}y^6}$

10. $\sqrt{\frac{x^2}{y^6}}$

11. $\sqrt{\frac{3a^3}{2x^{-5}}}$

12. $\sqrt[3]{\frac{a^{\frac{1}{2}}}{b^{\frac{3}{2}}}}$

13. $\sqrt{\frac{7a^{-4}x^{-8}}{8m^{-\frac{1}{2}}}}$

14. $\sqrt[5]{3^{10}x^{-5}y^{-2}}$

15. $\sqrt{a^3b^{-n}m^{-4}}$

16. $\sqrt[4]{\frac{3\,b^x\,c^{-y}}{m^zn^{-u}}}$

17. $\sqrt{a^2b^{-1}x^{-3}}$

155. Miscellaneous Exercises.

ILLUSTRATIVE EXERCISES

1. Simplify $\left(\dfrac{8x^{-3}}{27z^{-1}}\right)^{-\frac{2}{3}}$

Solution:

$$\begin{aligned}\left(\frac{8x^{-3}}{27z^{-1}}\right)^{-\frac{2}{3}} &= \frac{8^{-\frac{2}{3}}x^{2}}{27^{-\frac{2}{3}}z^{\frac{2}{3}}} && \text{(Law V)}\\ &= \frac{27^{\frac{2}{3}}x^{2}}{8^{\frac{2}{3}}z^{\frac{2}{3}}} && \text{(meaning of negative exponent)}\\ &= \frac{9x^{2}}{4z^{\frac{2}{3}}} && \text{(meaning of fractional exponent)}\end{aligned}$$

2. Multiply $x^{\frac{1}{2}}+a^{\frac{1}{2}}$ by $x^{\frac{1}{2}}-a^{\frac{1}{2}}$

Solution:

$$\begin{array}{lll} x^{\frac{1}{2}}+a^{\frac{1}{2}} & & \\ x^{\frac{1}{2}}-a^{\frac{1}{2}} & & \\ \hline x & +x^{\frac{1}{2}}a^{\frac{1}{2}} & \\ & -x^{\frac{1}{2}}a^{\frac{1}{2}} & -a \\ \hline x & & -a \end{array}$$

Problem 2 might have been solved by inspection. The product of the sum and difference of two numbers is to be found. Hence, the result is the difference of the squares of the two numbers. That is,

$$(x^{\frac{1}{2}}+a^{\frac{1}{2}})\,(x^{\frac{1}{2}}-a^{\frac{1}{2}})=(x^{\frac{1}{2}})^{2}-(a^{\frac{1}{2}})^{2}=x-a$$

3. $(x^{2}-x^{-2})^{2}=?$

Solution:

$$\begin{aligned}(x^{2}-x^{-2})^{2}&=(x^{2})^{2}-2(x^{2})\,(x^{-2})+(x^{-2})^{2}\\ &=x^{4}-2x^{0}+x^{-4}=x^{4}-2+x^{-4}\end{aligned}$$

In the following problems perform the indicated operations and simplify as far as possible:

1. $x^{3}\cdot x^{-2}$
3. $m^{\frac{2}{3}}\cdot m^{-\frac{1}{2}}$
5. $b^{\frac{3}{4}}\cdot b^{-\frac{1}{2}}$

2. $x^{-4}\cdot x^{3}\cdot x^{5}$
4. $y^{-2}\cdot y^{-3}\cdot y^{4}$
6. $x^{\frac{2}{3}}\cdot x^{\frac{1}{3}}\cdot x^{0}$

7. $x^{\frac{5}{3}} \cdot x^{-2} \cdot x^{\frac{1}{3}}$

8. $c^{-3} \cdot c^{\frac{4}{3}} \cdot c^{\frac{2}{3}}$

9. $8x^{-\frac{1}{2}} \cdot x^{\frac{1}{3}} \cdot x^{\frac{3}{2}}$

10. $5(a+b)^0+64^0$

11. $x^2 \div x^{-1}$

12. $x^4 \div x^{-5}$

13. $b^{-2} \div b$

14. $z^{\frac{3}{2}} \div z^{\frac{1}{2}}$

15. $\dfrac{m^{-2}}{m^{-1}}$

16. $\dfrac{r^{-3}}{r^{-5}}$

17. $s^{-\frac{4}{3}} \div s^{\frac{2}{3}}$

18. $12n^{-\frac{3}{4}} \div 3n^{-\frac{1}{2}}$

19. $\sqrt{\dfrac{2c^0}{c^4}}$

20. $\sqrt{\dfrac{25x^4}{m^{-2}}}$

21. $\sqrt[3]{\dfrac{a^{-3}}{x^{-6}}}$

22. $\sqrt{\dfrac{a^{-4}}{x^{-2}}}$

23. $\sqrt[3]{a^{\frac{3}{4}}x^{\frac{3}{2}}}$

24. $(3^2)^2$

25. $(3^2)^{-2}$

26. $(5^2)^{\frac{1}{2}}$

27. $(8^{-2})^{\frac{1}{3}}$

28. $(16^{\frac{1}{2}})^{-3}$

29. $(3x^2)^{-\frac{3}{2}}$

30. $(4a^2c^2)^{\frac{1}{2}}$

31. $(4^{-2})^{-1}$

32. $[(\frac{1}{2})^2]^2$

33. $(2x^{\frac{3}{2}})^{\frac{1}{2}}$

34. $(3x^2y^{-4})^0$

35. $x^{2a} \cdot x^{3a}$

36. $m^{2x} \cdot m^{-x}$

37. $4x^b \cdot 3x^{-2b}$

38. $(3c^{-1}d^{-1})^x$

39. $(x^{a-2})^3$

40. $(a+x^{-1})^2$

41. $(x^{\frac{1}{2}}-y^{\frac{1}{2}})^2$

42. $(m^{\frac{1}{3}}+n^{-\frac{1}{3}})^2$

43. $(z^{\frac{2}{3}}+3)^2$

44. $(e^x+e^{-x})^2$

45. $(x^0+x^{\frac{3}{4}})^2$

46. $(x^{-1}+1)^3$

47. $(x^2-a^{-2})^3$

48. $(m^{\frac{1}{2}}+n^{\frac{1}{2}})^2$

49. $(r^{\frac{2}{3}}-s^{\frac{1}{3}})^2$

50. $(b^2-c^2b) \div b^3$

51. $a^2x^{-3}y^2 \div a^4x^{-5}y^{-3}$

52. $(4r^2-2s^{-1})3r$

53. $(x^2-3ax)x^{-1}$

54. $(a+x^{-1})\ (a-x^{-1})$

55. $(m^{\frac{1}{3}}-3)\ (m^{\frac{1}{3}}+3)$

56. $(2z^{-2}-z^{-1}+1)^2$

57. $(m^{\frac{1}{3}}+1+m^{-\frac{1}{3}})^2$

58. $(m^{-4}+2m^{-2}n^2+n^4)^{\frac{1}{2}}$

59. $(x^{-2}+2+x^2)^{\frac{1}{2}}$.

Hint. $2=2 \cdot x^{-1} \cdot x$

60. $(x^3-4x^2+3x) \div x^4$

61. $(x-y) \div (x^{\frac{1}{2}}-y^{\frac{1}{2}})$

62. $(a^{\frac{1}{2}}-x^{-\frac{1}{2}})\ (a^{\frac{1}{2}}-x^{-\frac{1}{2}})$

63. $(c^{-2}-c^{-1}d+d^2)\ (c^{-1}+d)$

64. $(m^{\frac{2}{3}}-n^{\frac{2}{3}}) \div (m^{\frac{1}{3}}+n^{\frac{1}{3}})$

65. $\sqrt{x^{-2}+2x^{-1}y^{-1}+y^{-2}}$

66. $(x^{-2}+5x^{-1}+6)\div(x^{-1}+3)$

67. $(x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}})\ (x^{\frac{1}{3}}-y^{\frac{1}{3}})$

68. $(8a^{\frac{3}{4}}+4a^{\frac{1}{2}}b^{\frac{1}{2}}+2a^{\frac{1}{4}}b+b^{\frac{3}{2}})\ (2a^{\frac{1}{4}}-b^{\frac{1}{2}})$

69. $(x^4+7x^2+8-7x^{-2}+x^{-4})\div(x^2+5-x^{-2})$

70. Simplify $8^{\frac{1}{2}}+18^{\frac{1}{2}}-32^{\frac{1}{2}}$

Solution:

$$\begin{aligned} 8^{\frac{1}{2}}+18^{\frac{1}{2}}-32^{\frac{1}{2}} &= 4^{\frac{1}{2}}\cdot 2^{\frac{1}{2}}+9^{\frac{1}{2}}\cdot 2^{\frac{1}{2}}-16^{\frac{1}{2}}\cdot 2^{\frac{1}{2}} \\ &= 2\cdot 2^{\frac{1}{2}}+3\cdot 2^{\frac{1}{2}}-4\cdot 2^{\frac{1}{2}} \\ &= (2+3-4)2^{\frac{1}{2}} = 2^{\frac{1}{2}} \end{aligned}$$

71. Simplify $12^{\frac{1}{2}}+27^{\frac{1}{2}}-75^{\frac{1}{2}}$

72. Simplify $54^{\frac{1}{3}}+16^{\frac{1}{3}}-128^{\frac{1}{3}}+2^{\frac{1}{3}}$

156. Radicals. Two methods of expressing a root of a number are used. $\sqrt[3]{10}$ and $10^{\frac{1}{3}}$ both express the cube root of 10. The former is the older form and would probably not have been used had the fractional exponent been in use earlier. Sometimes the one and sometimes the other of these forms is preferable, and the student should be able to change readily from one to the other. In using fractional exponents it is usually best to use a prime number as base. For example, $\sqrt[3]{9}$ may be expressed as $9^{\frac{1}{3}}$ or $3^{\frac{2}{3}}$, but the latter is usually preferable.

ILLUSTRATIVE EXERCISES

Express $\sqrt[3]{25}$ by means of fractional exponents.

Solution:

$$\sqrt[3]{25} = \sqrt[3]{5^2} = 5^{\frac{2}{3}}$$

EXERCISES

Express with fractional exponents:

1. $\sqrt{5}$
2. $\sqrt[5]{16}$
3. $\sqrt[3]{16}$
4. $\sqrt[5]{27}$
5. $\sqrt[10]{64}$
6. $\sqrt{8}$
7. $\sqrt[7]{9}$
8. $\sqrt[4]{8}$
9. $\sqrt[3]{12}$
10. $\sqrt[4]{20}$
11. $\sqrt[4]{24}$
12. $\sqrt[5]{48}$
13. $\sqrt{abx}$
14. $\sqrt[3]{a^2x^4y}$
15. $\sqrt{m^3n^5}$
16. $\sqrt[4]{5b^2c^3d^5}$
17. $\sqrt[m]{a^3}$
18. $\sqrt[k]{cr}$
19. $\sqrt{6} = 6^{\frac{1}{2}} = 3^{\frac{1}{2}} \cdot 2^{\frac{1}{2}}$
20. $\sqrt[n]{a^2x^am^b}$

Express the following as radicals:

1. $2^{\frac{2}{3}}$
2. $3^{\frac{3}{4}}$
3. $5^{\frac{1}{3}}$
4. $2^{\frac{1}{3}} \cdot 3^{\frac{2}{3}}$
5. $5^{\frac{3}{4}}$
6. $6^{\frac{2}{5}}$
7. $a^{\frac{r}{s}}$
8. $c^{\frac{1}{x}}\, m^{\frac{2}{x}}$
9. $3^{\frac{1}{2}}a^{\frac{3}{2}}b^{\frac{a}{2}}$

When is a radical involving square root in its simplest form? (Section 129)

EXERCISES

Reduce the following radicals to their simplest forms:

1. $\sqrt{8}$
2. $\sqrt{50}$
3. $\sqrt{75}$
4. $\sqrt{84}$
5. $\sqrt{27}$
6. $\sqrt{98}$
7. $\sqrt{180}$
8. $\sqrt{243}$
9. $\sqrt{\frac{4}{5}}$
10. $\sqrt{\frac{7}{8}}$
11. $\sqrt{\frac{3}{4}}$
12. $\sqrt{\frac{5}{6}}$
13. $\sqrt{a^6b^3}$
14. $\sqrt{a^2b^3y^5}$
15. $\sqrt{\frac{a^5}{x}}$
16. $\sqrt{\frac{s}{x^2y^3}}$

A radical is in its simplest form when

I. The number under the radical is an integer.

II. The number under the radical sign contains no factor of which the indicated root can be taken.

III. The indices of the power and root involved are prime to each other.

ILLUSTRATIVE EXERCISES

1. Simplify $\sqrt[3]{\frac{2}{3}}$

Solution:

$$\sqrt[3]{\frac{2}{3}} = \sqrt[3]{\frac{18}{27}} = \frac{\sqrt[3]{18}}{\sqrt[3]{27}} = \frac{\sqrt[3]{18}}{3} = \frac{1}{3}\sqrt[3]{18}$$

2. Simplify $\sqrt[4]{48}$

Solution:

$$\sqrt[4]{48} = \sqrt[4]{16} \cdot \sqrt[4]{3} = 2\sqrt[4]{3}$$

3. Simplify $\sqrt[6]{27}$

Solution:

$$\sqrt[6]{27} = \sqrt[6]{3^3} = 3^{\frac{3}{6}} = 3^{\frac{1}{2}} = \sqrt{3}$$

EXERCISES

Reduce the following to their simplest forms:

1. $\sqrt{54}$
2. $\sqrt[3]{54}$
3. $\sqrt[4]{32}$
4. $\sqrt[3]{\frac{1}{2}}$
5. $\sqrt[3]{\frac{8}{9}}$
6. $\sqrt[4]{\frac{3}{2}}$
7. $\sqrt[6]{\frac{1}{2}}$
8. $\sqrt[6]{\frac{5}{2}}$
9. $\sqrt[5]{\frac{2}{3}}$
10. $\sqrt[3]{\frac{5}{9}}$
11. $\sqrt[4]{96}$
12. $\sqrt[3]{a^3x^2}$
13. $\sqrt[4]{a^5b}$
14. $\sqrt[3]{m^6n^4p^3}$

15. $\sqrt[6]{8a^3}$

16. $\sqrt[4]{a^6b^5}$

17. $\sqrt[6]{9a^6x^2}$

18. $\sqrt[3]{\frac{2}{9a}}$

19. $\sqrt[3]{72}$

20. $\sqrt[4]{\frac{96}{x^2}}$

21. $\sqrt[3]{250bd}$

Sometimes it is desirable to introduce a rational factor under the radical sign.

ILLUSTRATIVE EXERCISES

1. $2\sqrt{3} = \sqrt{4} \cdot \sqrt{3} = \sqrt{12}$

2. $3a\sqrt[3]{4} = \sqrt[3]{27a} \cdot \sqrt[3]{4} = \sqrt[3]{108a^3}$

How may a rational factor be introduced under the radical sign?

EXERCISES

In the following problems introduce the rational factor under the radical sign:

1. $5\sqrt{2}$
2. $6\sqrt{3}$
3. $3\sqrt{x}$
4. $2\sqrt[3]{5}$
5. $a\sqrt{x}$
6. $4\sqrt{7}$
7. $2\sqrt[3]{c}$
8. $\frac{1}{2}\sqrt[3]{4}$
9. $b\sqrt{a^2b}$
10. $3\sqrt[3]{2x}$
11. $x\sqrt[3]{x}$
12. $s\sqrt[4]{a}$
13. $ay\sqrt[3]{a^2y}$
14. $\frac{2}{5}\sqrt{4}$
15. $\frac{1}{2}\sqrt[4]{2}$
16. $a^2b\sqrt{ab}$
17. $\frac{2}{a}\sqrt{\frac{a}{2}}$
18. $\frac{3}{2}\sqrt[3]{xy}$
19. $\frac{2x}{5}\sqrt{25x}$
20. $\frac{1}{a-b}\sqrt{a^2-b^2}$

157. Radicals having the same index and the same number under the radical sign are called **similar**. For example, $\sqrt{5}$, $2\sqrt{5}$, and $\frac{1}{2}\sqrt{5}$ are similar radicals. $\sqrt[3]{6}$. $\sqrt{6}$, and $\sqrt[3]{5}$ are not similar.

EXERCISES

Which of the following sets are similar radicals?

1. $\sqrt{2},\ 3\sqrt{2}$
2. $\sqrt{5},\ \sqrt[3]{5}$
3. $\sqrt[3]{4},\ \sqrt[4]{4}$
4. $\sqrt[3]{a^2x},\ \sqrt[3]{ax^2}$
5. $\sqrt{ax},\ 3z\sqrt{ax}$
6. $\sqrt[3]{x^2y},\ 4\sqrt[3]{xy}$
7. $\sqrt[3]{abc},\ \sqrt{abc}$

158. Addition and Subtraction of Radicals.

In order that two or more radicals may be combined by addition or subtraction into one term, the radicals must be similar.

Hence, to add or subtract radicals:

Reduce each radical to its simplest form and add or subtract the coefficients of the similar radicals.

The addition or subtraction of dissimilar radicals can only be indicated.

ILLUSTRATIVE EXERCISES

1. Add $\sqrt{3}$, $2\sqrt{3}$, and $\sqrt{75}$.

Solution:

$$\sqrt{3}+2\sqrt{3}+\sqrt{75}=\sqrt{3}+2\sqrt{3}+5\sqrt{3}$$
$$=(1+2+5)\sqrt{3}=8\sqrt{3}$$

2. Combine into one term $2\sqrt{\frac{3}{5}}-\sqrt{\frac{5}{3}}+3\sqrt{\frac{1}{15}}$

Solution:

$$\sqrt{\frac{3}{5}} = \sqrt{\frac{15}{25}} = \frac{1}{5}\sqrt{15}$$

$$\sqrt{\frac{5}{3}} = \sqrt{\frac{15}{9}} = \frac{1}{3}\sqrt{15}$$

$$\sqrt{\frac{1}{15}} = \sqrt{\frac{15}{225}} = \frac{1}{15}\sqrt{15}$$

Hence, $2\sqrt{\frac{3}{5}} - \sqrt{\frac{5}{3}} + 3\sqrt{\frac{1}{15}} = \left(\frac{2}{5} - \frac{1}{3} + \frac{1}{5}\right)\sqrt{15}$

$$= \frac{4}{15}\sqrt{15}$$

EXERCISES

Combine each of the following sets of radicals and reduce each result to its simplest form:

1. $2\sqrt{5} + 3\sqrt{5} + \sqrt{5}$
2. $3\sqrt{2} - 2\sqrt{2} + 5\sqrt{2}$
3. $2\sqrt[3]{x} + 3\sqrt[3]{x} - \sqrt[3]{x}$
4. $\sqrt{150} + \sqrt{96} - \sqrt{54}$
5. $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{8}} + \sqrt{\frac{1}{18}}$
6. $\sqrt{a^2 - b^2} + \sqrt{\frac{a-b}{a+b}}$
7. $\sqrt{12} - \sqrt{3}$
8. $2\sqrt{40} - \sqrt{90}$
9. $3\sqrt[3]{4} - \sqrt[3]{4}$
10. $\sqrt{50} + \sqrt{98}$
11. $\sqrt{45} + \sqrt{80}$
12. $\sqrt{112} - \sqrt{28}$
13. $2\sqrt{2} - \sqrt{\frac{1}{2}}$
14. $3\sqrt{3} + 2\sqrt{\frac{1}{3}}$
15. $\sqrt{\frac{2}{3}} + \sqrt{\frac{3}{2}}$
16. $\sqrt{a^2x} + \sqrt{b^2x}$
17. $\sqrt{a^2x} + \sqrt{x}$
18. $\sqrt[3]{16} + \sqrt[3]{54}$

19. $\sqrt[3]{192} - \sqrt[3]{81}$

20. $\sqrt[3]{x^4} - \sqrt[3]{x}$

21. $\sqrt{\frac{5}{3}} + 2\sqrt{\frac{3}{5}}$

22. $2\sqrt{\frac{3}{8}} - 3\sqrt{\frac{3}{2}}$

23. $\sqrt[3]{\frac{3}{4}} + \sqrt[3]{48}$

24. $(x+y)\sqrt{\frac{x-y}{x+y}} + (x-y)\sqrt{\frac{x+y}{x-y}}$

159. Multiplication of Radicals.

Radicals having the same index are multiplied by multiplying the numbers under the radical signs. (See law VI.)

ILLUSTRATIVE EXERCISES

1. Multiply $\sqrt{3}$ by $\sqrt{5}$

Solution:

$$\sqrt{3} \cdot \sqrt{5} = \sqrt{3 \cdot 5} = \sqrt{15}$$

2. Multiply $2\sqrt[3]{4a^2x}$ by $3\sqrt[3]{2a}$

Solution:

$$2\sqrt[3]{4a^2x} \cdot 3\sqrt[3]{2a} = 2 \cdot 3\sqrt[3]{(4a^2x)(2a)}$$
$$= 6\sqrt[3]{8a^3x} = 12a\sqrt[3]{x}$$

EXERCISES

Carry out the following multiplications and reduce each result to its simplest form:

1. $\sqrt{2} \cdot \sqrt{5}$
2. $\sqrt{2} \cdot \sqrt{6}$
3. $\sqrt{3} \cdot \sqrt{7}$
4. $\sqrt{6} \cdot \sqrt{20}$
5. $\sqrt{14} \cdot \sqrt{21}$
6. $\sqrt{3} \cdot \sqrt{15}$
7. $\sqrt{5a} \cdot \sqrt{3a}$
8. $\sqrt{8c} \cdot \sqrt{7c}$
9. $\sqrt[4]{6a^3} \cdot \sqrt[4]{2ab^3}$
10. $2\sqrt{22} \cdot 3\sqrt{6}$

11. $\sqrt[3]{4} \cdot \sqrt[3]{6}$

12. $\sqrt[3]{10} \cdot \sqrt[3]{20}$

13. $\sqrt[3]{12} \cdot \sqrt[3]{18}$

14. $\sqrt[3]{7} \cdot \sqrt[3]{6}$

15. $\sqrt{\frac{a}{b}} \cdot \sqrt{\frac{b}{a}}$

16. $\sqrt{\frac{a}{x}} \cdot \sqrt{3}$

17. $\sqrt{\frac{2}{3}} \cdot \sqrt{6}$

18. $\sqrt{\frac{3}{5}} \cdot \sqrt{\frac{4}{15}}$

19. $\sqrt{.2} \cdot \sqrt{.32}$

20. $\sqrt{.08} \cdot \sqrt{.18}$

Radicals not having the same index may be reduced to radicals having the same index and multiplied as above.

We assume $a^{\frac{r}{s}} = a^{\frac{kr}{ks}}$

ILLUSTRATIVE EXERCISES

1. Multiply $\sqrt[3]{2}$ by $\sqrt{3}$

Solution:

$$\sqrt[3]{2} = 2^{\frac{1}{3}} = 2^{\frac{2}{6}} = \sqrt[6]{4}$$

$$\sqrt{3} = 3^{\frac{1}{2}} = 3^{\frac{3}{6}} = \sqrt[6]{27}$$

Hence, $\sqrt[3]{2} \cdot \sqrt{3} = \sqrt[6]{4} \cdot \sqrt[6]{27} = \sqrt[6]{108}$

2. Multiply $\sqrt[3]{25}$ by $\sqrt{10}$

Solution:

$$\sqrt[3]{25} = 5^{\frac{2}{3}}$$

$$\sqrt{10} = 5^{\frac{1}{2}} \cdot 2^{\frac{1}{2}}$$

Hence, $\sqrt[3]{25} \cdot \sqrt{10} = 5^{\frac{2}{3}} \cdot 5^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} = 5^{\frac{7}{6}} \cdot 2^{\frac{1}{2}} = 5 \cdot 5^{\frac{1}{6}} \cdot 2^{\frac{3}{6}}$

$$= 5\sqrt[6]{5 \cdot 2^3} = 5\sqrt[6]{40}$$

The exponents of 5 in the factors are added to obtain the exponent of 5 in the product.

Problem 2 might have been solved by reducing the radicals to a common index and then multiplying the numbers under the radical signs, but unless prime numbers had been used as bases, the rational factor 5 might have been overlooked.

3. Multiply $2\sqrt{3}+\sqrt{6}$ by $\sqrt{6}-\sqrt{2}$

Solution:

$$\begin{array}{rl}
2\sqrt{3}+\sqrt{6} \qquad\qquad\qquad\qquad & \\
\sqrt{6}-\sqrt{2} \qquad\qquad\qquad\qquad & \\
\hline
2\sqrt{18}+\sqrt{(6)^2} \qquad\qquad\qquad\qquad & \text{(multiplying by } \sqrt{6}) \\
-2\sqrt{6}-\sqrt{12} & \text{(multiplying by } \sqrt{2}) \\
\hline
2\sqrt{18}+\sqrt{(6)^2}-2\sqrt{6}-\sqrt{12} & \\
=6\sqrt{2}+6-2\sqrt{6}-2\sqrt{3} \qquad &
\end{array}$$

EXERCISES

Multiply and reduce results to the simplest form:

1. $\sqrt{2}$ by $\sqrt[3]{3}$
2. $\sqrt[3]{4}$ by $\sqrt{6}$
3. $\sqrt[3]{9}$ by $\sqrt{3}$
4. $\sqrt[3]{9}$ by $\sqrt{15}$
5. $\sqrt[3]{x^2}$ by $\sqrt{ax}$
6. $\sqrt[3]{25}$ by $\sqrt{10}$
7. $\sqrt[3]{a^2cx}$ by $\sqrt{acx}$
8. $\sqrt{x^2yz}$ by $\sqrt[3]{y^2z^2}$
9. $(\sqrt{5}+\sqrt{3})^2$
10. $(\sqrt{7}-\sqrt{5})^2$
11. $(\sqrt{a}-\sqrt{x})^2$
12. $(\sqrt{6}+\sqrt{14})^2$
13. $(2\sqrt{3}+\sqrt{8})^2$
14. $(\sqrt{a}-\sqrt{k})^2$
15. $(3\sqrt{8}-2\sqrt{3})^2$
16. $(\sqrt{10}+\sqrt{18}+\sqrt{6})\sqrt{2}$
17. $(\sqrt{6}+\sqrt{54}+\sqrt{24})\sqrt{3}$
18. $(\sqrt{10}-\sqrt{15}+\sqrt{20})\sqrt{5}$

19. $(\sqrt[3]{4}+\sqrt[3]{12}+\sqrt[3]{20})\sqrt[3]{2}$

20. $(\sqrt{2a}+\sqrt{8a}-\sqrt{18a})\sqrt{2a}$

21. $(\sqrt[3]{a^2b^2}-\sqrt[3]{b^2c^2}+\sqrt[3]{c^2a^2})\sqrt[3]{abc}$

22. $\left(\sqrt{\frac{3}{2}}+\sqrt{\frac{3}{8}}-\sqrt{\frac{3}{50}}\right)\sqrt{\frac{1}{2}}$

23. $(\sqrt{7}+\sqrt{2})(\sqrt{7}-\sqrt{2})$

24. $(\sqrt{10}-\sqrt{3})(\sqrt{10}+\sqrt{3})$

25. $(\sqrt{a}+\sqrt{x})(\sqrt{a}-\sqrt{x})$

26. $(2\sqrt{5}-\sqrt{3})(\sqrt{15}+\sqrt{6})$

27. $(\sqrt{10m}+\sqrt{6n})(\sqrt{2n}-\sqrt{5m})$

28. $(\sqrt[3]{x^2}-\sqrt[3]{ax}+\sqrt[3]{a^2})(\sqrt[3]{x}+\sqrt[3]{a})$

160. Division of Radicals. Radicals having the same index are divided by dividing the number under the radical sign in the dividend by the number under the radical sign in the divisor.

ILLUSTRATIVE EXERCISES

1. Divide $\sqrt{6}$ by $\sqrt{2}$

Solution:

$$\sqrt{6}\div\sqrt{2}=\sqrt{6\div 2}=\sqrt{3}$$

2. Divide $\sqrt[3]{7}$ by $\sqrt[3]{4}$

Solution:

$$\sqrt[3]{7}\div\sqrt[3]{4}=\sqrt[3]{7\div 4}=\sqrt[3]{\frac{7}{4}}=\frac{1}{2}\sqrt[3]{14}$$

EXERCISES

Divide and reduce the results to the simplest form:

1. $\sqrt{12} \div \sqrt{3}$
2. $\sqrt{10} \div \sqrt{5}$
3. $\sqrt{14} \div \sqrt{7}$
4. $\sqrt{18} \div \sqrt{6}$
5. $\sqrt{22} \div \sqrt{2}$
6. $\sqrt{30} \div \sqrt{5}$
7. $3\sqrt{35} \div \sqrt{7}$
8. $6\sqrt{42} \div 2\sqrt{14}$
9. $\sqrt[3]{12} \div \sqrt[3]{4}$
10. $\sqrt[3]{36} \div \sqrt[3]{6}$
11. $\sqrt{14} \div \sqrt{3}$
12. $\sqrt{a^3x} \div \sqrt{a^2x}$
13. $x^2\sqrt{m^3n^5} \div \sqrt{m^3n^3}$
14. $12\sqrt{x^5y^2} \div 6\sqrt{x^2y}$
15. $6\sqrt{a^2-b^2} \div 3\sqrt{a-b}$
16. $\sqrt{42} \div \sqrt{28}$
17. $\sqrt{35} \div \sqrt{10}$
18. $\sqrt[3]{24} \div \sqrt[3]{16}$
19. $\sqrt{56} \div \sqrt{18}$
20. $\sqrt{38} \div \sqrt{6}$
21. $\sqrt{m} \div \sqrt{n}$
22. $\sqrt{ax} \div \sqrt{bx}$
23. $\sqrt{c^2k} \div \sqrt{c^3k^2}$
24. $\sqrt{x^3-y^3} \div \sqrt{x-y}$

Radicals having different indices may be reduced to radicals having the same index and divided as above.

ILLUSTRATIVE EXERCISE

Divide $\sqrt[3]{18}$ by $\sqrt{6}$

Solution:

$$\frac{\sqrt[3]{18}}{\sqrt{6}} = \frac{\sqrt[3]{3^2 \cdot 2}}{\sqrt{3 \cdot 2}} = \frac{3^{\frac{2}{3}} \cdot 2^{\frac{1}{3}}}{3^{\frac{1}{2}} \cdot 2^{\frac{1}{2}}} = \frac{3^{\frac{1}{6}}}{2^{\frac{1}{6}}} = \sqrt[6]{\frac{3}{2}} = \frac{1}{2}\sqrt[6]{96}$$

EXERCISES

Simplify the following:

1. $\sqrt[3]{4} \div \sqrt{2}$
2. $\sqrt[3]{25} \div \sqrt{5}$
3. $\sqrt[3]{36} \div \sqrt{6}$
4. $\sqrt[4]{8} \div \sqrt{2}$

5. $\sqrt{50} \div \sqrt[3]{10}$

6. $3\sqrt[3]{75} \div \sqrt{5}$

7. $5\sqrt[3]{98} \div 2\sqrt{14}$

8. $3\sqrt[3]{21} \div 2\sqrt{35}$

9. $4\sqrt{10} \div 5\sqrt[3]{15}$

10. $\sqrt{ax} \div \sqrt[4]{ax^2}$

11. $\sqrt[4]{c^3k} \div \sqrt{ck}$

12. $\sqrt[3]{a^2b} \div \sqrt[4]{a^2b^3}$

161. Rationalization. Rationalization is the process of multiplying a radical expression by such a number that the product will be rational.

ILLUSTRATIVE EXERCISE

Rationalize $\sqrt[3]{4}$

Solution:

$$\sqrt[3]{4} \cdot \sqrt[3]{2} = \sqrt[3]{8} = 2$$

The $\sqrt[3]{2}$ is called a **rationalizing factor** of $\sqrt[3]{4}$.

If the denominator of a fraction is irrational, it is necessary to make it rational in order that the expression may be in its simplest form. The process is called **rationalizing the denominator.**

ILLUSTRATIVE EXERCISES

1. Rationalize the denominator of $\dfrac{\sqrt{5}}{\sqrt{3}}$.

Solution:

Multiply numerator and denominator by $\sqrt{3}$.

$$\frac{\sqrt{5}}{\sqrt{3}} = \frac{\sqrt{5} \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \frac{\sqrt{15}}{3}$$

2. Rationalize the denominator of $\dfrac{1}{\sqrt[3]{4}}$.

Solution:

Multiply numerator and denominator by $\sqrt[3]{2}$.

$$\frac{1}{\sqrt[3]{4}} = \frac{\sqrt[3]{2}}{\sqrt[3]{4} \cdot \sqrt[3]{2}} = \frac{\sqrt[3]{2}}{\sqrt[3]{8}} = \frac{\sqrt[3]{2}}{2}$$

Rationalize the denominator in each of the following:

1. $\dfrac{\sqrt{6}}{\sqrt{5}}$
2. $\dfrac{\sqrt{7}}{\sqrt{3}}$
3. $\dfrac{\sqrt{10}}{\sqrt{7}}$
4. $\dfrac{\sqrt{5}}{\sqrt{11}}$
5. $\dfrac{\sqrt{22}}{\sqrt{12}}$
6. $\dfrac{\sqrt{45}}{\sqrt{6}}$
7. $\dfrac{\sqrt{x}}{\sqrt{a}}$
8. $\dfrac{\sqrt{cb}}{\sqrt{m}}$
9. $\dfrac{\sqrt{a-b}}{\sqrt{a+b}}$
10. $\dfrac{1}{\sqrt[4]{9}}$
11. $\dfrac{2}{\sqrt[3]{2}}$
12. $\dfrac{5}{\sqrt[3]{25}}$
13. $\dfrac{3}{\sqrt[4]{27}}$
14. $\dfrac{\sqrt[3]{3}}{\sqrt[3]{6}}$
15. $\dfrac{3}{\sqrt{8}}$
16. $\dfrac{ax}{\sqrt[3]{a^2x}}$
17. $\dfrac{\sqrt[3]{cy}}{\sqrt[3]{c^2y}}$
18. $\dfrac{1}{\sqrt[3]{x+y}}$
19. $\dfrac{(x+y)\sqrt{x-y}}{\sqrt{x+y}}$
20. $\dfrac{(x+y)(x-y)}{(\sqrt{x-y})(\sqrt{x+y})}$
21. $\dfrac{(x+y)\sqrt[3]{x-y}}{\sqrt[3]{(x+y)^2}}$

In problem 2, section 155, it was shown that products of the form $(\sqrt{x}+\sqrt{a})(\sqrt{x}-\sqrt{a})$ are rational. For example,

$$(\sqrt{x}+\sqrt{a})(\sqrt{x}-\sqrt{a}) = x - a$$

If the denominator of a fraction is a binomial whose terms are either both square roots or one rational and the other a square root, it may be rationalized by multiplying both terms of the fraction by the denominator with the sign between the terms changed.

Hear their Wings flap

ILLUSTRATIVE EXERCISE

Rationalize the denominator of $\frac{3}{\sqrt{5}+\sqrt{3}}$.

Solution:

$$\frac{3}{\sqrt{5}+\sqrt{3}}=\frac{3(\sqrt{5}-\sqrt{3})}{(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})}$$

$$=\frac{3\sqrt{5}-3\sqrt{3}}{5-3}=\frac{3\sqrt{5}-3\sqrt{3}}{2}$$

$$=\frac{3}{2}(\sqrt{5}-\sqrt{3})$$

EXERCISES

Change the following fractions to equivalent fractions having rational denominators:

1. $\frac{1}{\sqrt{3}-\sqrt{2}}$
2. $\frac{2}{2-\sqrt{3}}$
3. $\frac{6}{\sqrt{7}-\sqrt{5}}$
4. $\frac{4}{7-\sqrt{10}}$
5. $\frac{7}{7-\sqrt{30}}$
6. $\frac{2+\sqrt{3}}{\sqrt{5}-\sqrt{2}}$
7. $\frac{\sqrt{3}+\sqrt{5}}{\sqrt{3}-\sqrt{5}}$
8. $\frac{\sqrt{7}-\sqrt{5}}{\sqrt{7}+\sqrt{5}}$
9. $\frac{3}{\sqrt{x}-4}$
10. $\frac{6}{\sqrt{a}+\sqrt{3}}$
11. $\frac{a}{\sqrt{a}+\sqrt{x}}$
12. $\frac{c}{\sqrt{y}-\sqrt{x}}$
13. $\frac{x}{\sqrt{x+2}-\sqrt{x}}$
14. $\frac{x-2}{\sqrt{x+2}-\sqrt{x-2}}$

162. Equations Containing Radicals. We are now in a position to solve many equations containing radicals.

ILLUSTRATIVE EXERCISES

1. Solve $\sqrt{x-3}+5=8$

Solution:

Subtracting 5, $\sqrt{x-3}=3$
Squaring both members, $x-3=9$
Whence, $x=12$

Check: $\sqrt{12-3}+5=3+5=8$

2. Solve $17-\sqrt[3]{x}=12$

Solution:

Subtracting 17, $-\sqrt[3]{x}=-5$
Cubing both members, $-x=-125$
Whence, $x=125$

Check: $17-\sqrt[3]{125}=17-5=12$

3. Solve $\sqrt{x+5}-3=\sqrt{x+14}-2$

Solution:

Adding 3 to each member,

$$\sqrt{x+5}=\sqrt{x+14}+1$$

Squaring both members,

$$x+5=x+14+2\sqrt{x+14}+1$$

Collecting similar terms, $-10=2\sqrt{x+14}$
Dividing by 2, $-5=\sqrt{x+14}$
Squaring, $25=x+14$
Whence, $x=11$

Check: $\sqrt{11+5}-3=\sqrt{11+14}-2$

$4-3=5-2$

The result does not check. If, however, we use the negative value of the radical in each case it will check, since

$$-4-3=-5-2$$

Since it has been agreed that the radical shall denote only the positive square root unless otherwise stated, the solution found in this case is not a solution of the original equation. This frequently happens in the solution of this class of equations, and it is, therefore, necessary to check results carefully.

To solve an equation containing radical terms, proceed as follows:

I. Make one of the radical terms, usually the most complicated, one member of the equation.

II. Raise both members to the power necessary to remove this radical.

III. Continue this process until all radicals are removed.

IV. Solve the resulting equation.

EXERCISES

Solve the following equations and check:

1. $\sqrt{x} = 3$
2. $\sqrt[3]{x} = 4$
3. $\sqrt{x} + 3 = 6$
4. $\sqrt{x+4} = 3$
5. $\sqrt{x+6} = 4$
6. $\sqrt[3]{3x+3} = 3$
7. $\sqrt{x} - 2 = 3$
8. $2\sqrt{x} + 1 = 7$
9. $\sqrt{3x-2} + \sqrt{x+3} = \sqrt{8x+1}$
10. $\sqrt{x+3} = \sqrt{4x-15}$
11. $\sqrt[3]{x+4} = \sqrt[3]{x^2-8}$
12. $\sqrt{y} + 2 = \sqrt{2y+7}$
13. $\sqrt{m+4} = \sqrt{m-1} + 1$
14. $\sqrt{3r+1} = 4\sqrt{r-4}$
15. $\sqrt{3x+7} - \sqrt{x+3} = 2$
16. $2\sqrt{x} = \sqrt{x^2-7x+18}$

17. $2\sqrt{2x+3}+3\sqrt{x-2}=1$

18. $\dfrac{\sqrt{x}}{\sqrt{2x-3}}=\dfrac{\sqrt{2x+3}}{\sqrt{x}}$

19. $3\sqrt{z}=2\sqrt{z+5}$

20. $\dfrac{\sqrt{3x-2}}{\sqrt{10-x}}=5$

MISCELLANEOUS EXERCISES

163. Reduce each of the following to its simplest form:

1. $\sqrt{28}$
2. $3\sqrt{24}$
3. $4\sqrt{250}$
4. $3b\sqrt{12c^3d^4}$
5. $5\sqrt[3]{40a^3b^3c^3}$
6. $3\sqrt[3]{375}$
7. $3b\sqrt[4]{64a^4bc^5}$
8. $\sqrt{(a+b)^2c}$
9. $\sqrt{(x-y)^2z^2w}$
10. $\sqrt[m]{a^mb}$
11. $5\sqrt{\dfrac{1}{6}}$
12. $4\sqrt{\dfrac{1}{2}}$
13. $4\sqrt{\dfrac{5}{8}}$
14. $\sqrt{\dfrac{4}{7}}$
15. $3\sqrt[3]{\dfrac{1}{9}}$
16. $\dfrac{1}{2}\sqrt[4]{\dfrac{1}{2}}$
17. $\dfrac{1}{6}\sqrt{\dfrac{36}{5}}$
18. $c\sqrt{\dfrac{a}{c}}$
19. $\sqrt[3]{\dfrac{ab}{cd}}$
20. $\sqrt{\dfrac{4a}{b}}$
21. $\sqrt[4]{25}$
22. $\sqrt[4]{36}$
23. $\sqrt[6]{9}$
24. $\sqrt{27}$
25. $\sqrt[6]{a^3b^3}$
26. $\sqrt[4]{49a^2}$
27. $\sqrt[6]{125z^3}$
28. $\sqrt[10]{100z^2}$
29. $\sqrt[4]{\dfrac{1}{a^2}}$
30. $\sqrt[2n]{a^2}$

In the following examples, perform the indicated operations and simplify the results:

31. $2\sqrt{5}+3\sqrt{5}$
32. $\sqrt{2}+\sqrt{18}$
33. $\sqrt{3}-\sqrt{\dfrac{1}{3}}$
34. $\sqrt{\dfrac{2}{3}}-\sqrt{6}$
35. $\sqrt{2}+\sqrt{50}-\sqrt{72}$

36. $\sqrt{\frac{2}{3}}+\sqrt{\frac{3}{2}}-\frac{1}{2}\sqrt{6}$

37. $\sqrt[4]{x^2y^3}\div\sqrt{xy}$

38. $\frac{1}{2}\sqrt{ab}-\sqrt{\frac{a}{b}}+\sqrt{\frac{1}{ab}}$

39. $\sqrt{5a}\cdot\sqrt{10a}$

40. $\sqrt[3]{45}\cdot\sqrt[3]{18}$

41. $\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{3}}$

42. $\sqrt{\frac{3}{5}}\cdot\sqrt[3]{\frac{5}{6}}$

43. $\sqrt[3]{22}\div\sqrt{11}$

44. $\sqrt{3}\div\sqrt{5}$

45. $(\sqrt{2}+3)^2$

46. $\sqrt{a}+\sqrt{b}+\sqrt{a^2}+\sqrt{b^2}$

47. $(\sqrt{11}-\sqrt{13}+\sqrt{7})\sqrt{3}$

48. $(\sqrt{2}-\sqrt{5}+\sqrt{10})\sqrt{10}$

49. $(\sqrt{3}-\sqrt{5}+\sqrt{10})\sqrt{15}$

50. $(\sqrt[3]{100}-\sqrt[3]{25}+\sqrt[3]{4})\sqrt[3]{10}$

51. $(\sqrt{3}+4)(\sqrt{2}+3)$

52. $(\sqrt{6}-5)(\sqrt{3}+5)$

53. $(\sqrt[3]{2}+\sqrt[3]{3})\ (\sqrt[3]{4}+\sqrt[3]{5})$

54. $\left(\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}+\sqrt{\frac{1}{4}}\right)\sqrt{12}$

55. $(\sqrt{2}+\sqrt{8}-\sqrt{21})\div\sqrt{2}$

56. $\left(2\sqrt{\frac{1}{2}}-8\sqrt[3]{\frac{1}{4}}-4\right)\div\sqrt[3]{4}$

57. $(3\sqrt{a}-2\sqrt{b})^2$

58. $(\sqrt{7}-\sqrt{13})(\sqrt{7}+\sqrt{13})$

59. $(\sqrt{15}+\sqrt{14})(\sqrt{15}-\sqrt{14})$

60. $\left(\sqrt{\frac{2}{3}}+\sqrt{\frac{1}{2}}\right)\left(\sqrt{\frac{2}{3}}-\sqrt{\frac{1}{2}}\right)$

Russell Hoyt

Ralph Burgett

Rationalize the denominator in each of the following:

61. $\dfrac{1}{\sqrt{6}}$

62. $\dfrac{4}{\sqrt[3]{4}}$

63. $\dfrac{4\sqrt{3}+6}{2-\sqrt{3}}$

64. $\dfrac{\sqrt{8}-3}{4+\sqrt{3}}$.

65. $\dfrac{2\sqrt{3}-3\sqrt{2}}{3\sqrt{6}-2\sqrt{2}}$

66. $\dfrac{6}{\sqrt{x+6}-\sqrt{x-6}}$

67. Solve for x: $\sqrt{2x+1}+\sqrt{x+5}=6$

68. Solve for y: $2+\sqrt{3y-2}-\sqrt{8y}=0$

69. Solve for x: $\sqrt{5+2x}=\sqrt{3+x}+\sqrt{2+x}$

SUMMARY

SECTION
144. Review of positive integral exponents.
145. Extension of the index laws.
146. Zero exponent.
147. Negative exponent.
148. Fractional exponent.
149. Exercises involving the application of law I.
150. Exercises involving the application of law II.
151. Exercises involving the application of law III.
152. Exercises involving the application of law IV.
153. Exercises involving the application of law V.
154. Exercises involving the application of law VI.
155. Miscellaneous exercises.
156. Radicals. Simplification of radicals
157. Definition of similar radicals.
158. Addition and subtraction of radicals.
159. Multiplication of radicals.
160. Division of radicals.
161. Rationalization.
162. Equations containing radicals.
163. Miscellaneous exercises.

CHAPTER XXIII

RATIO, VARIATION, AND PROPORTION

164. Ratio.

The quotient obtained by dividing one number by another is called the **ratio** of the first number to the second. A ratio is usually written in the form of a fraction. Thus, the ratio of 2 to 5 is written $\frac{2}{5}$. The ratio of a to b is written $\frac{a}{b}$. These ratios may also be written $2:5$ and $a:b$, respectively.

Ratios are fractions and subject to the laws of fractions.

In the ratio $\frac{a}{b}$, the first term, a, is called the **antecedent** and the second term, b, is called the **consequent** of the ratio.

EXERCISES

1. What is the ratio of 4 to 2?

Solution:

Since the ratio of 4 to 2 is the quotient of 4 divided by 2, we have

$$\tfrac{4}{2} = 2$$

Hence, the ratio of 4 to 2 is 2.

2. What is the ratio of 12 to 8? $\frac{12}{8} = \frac{3}{2}$.

Find the ratio of:

3. 15 to 5
4. 18 to 30
5. 45 to 18
6. 27 to 36
7. a^3 to a^2
8. a^2x^3 to x^3
9. $a^2 - x^2$ to $a + x$
10. $a^3 - x^3$ to $a^2 - x^2$
11. $a^2 - 5a + 6$ to $a^2 - 4a + 3$
12. $m + n$ to $m^3 + n^3$
13. $a(x + y)^2$ to $x^2 - 2xy - 3y^2$
14. $6x^2 - 17x - 3$ to $x - 3$

15. If for every \$7 B has, A has \$8, what is the ratio of the number of dollars B has to the number A has?

16. The circumference of a circle whose diameter is 10 inches is 31.416 inches. What is the approximate ratio of the circumference of the circle to its diameter?

17. If in problem 15 A and B together have \$75, how much does A have? How much does B have?

Suggestion. Let $8x$ equal the number of dollars A has. Then $7x$ equals the number of dollars B has.

18. The sum of two numbers is 56 and their ratio is $\frac{3}{4}$. What are the numbers?

Suggestion. Let $3x$ equal the smaller number. What equals the larger number?

19. The sum of two numbers is 77 and the ratio of the larger to the smaller is $\frac{9}{2}$. What are the numbers?

20. Two numbers differ by 70 and the ratio of the smaller to the larger is $\frac{2}{7}$. What are the numbers?

21. The ratio of the length of a rectangle to its width is $\frac{8}{3}$ and the distance around the rectangle is 220 feet. What are the dimensions of the rectangle?

22. The tens' digit of a number is two greater than the ones' digit. The ratio of the number to the number obtained by reversing the order of the digits is $\frac{7}{4}$. What is the number?

23. The ratio of a man's age to his son's age is 6. Five years from now the ratio of the father's age to his son's age will be $\frac{7}{2}$. What is the age of each?

165. Variation.

If two numbers change so that their ratio is always the same, one number is said to **vary directly** as the other.

If $\frac{y}{x} = m$, that is, $y = mx$, y is said to vary directly as x.

If $\frac{y}{x} = 5$, then $y = 5x$. That is, to satisfy this equation, y must always be 5 times as large as x. If $x = 3$, $y = 5 \cdot 3 = 15$.

If any two of the numbers y, x, and m, are known, the third can be found.

EXERCISES

1. If y varies directly as x and $y = 12$ when $x = 4$, what is the ratio of the two numbers?

2. If the ratio of a to b is always 4, what is the value of a when $b = 5$?

Solution:

By the conditions of the problem, $\frac{a}{b} = 4$

Substituting for b its value 5, $\frac{a}{5} = 4$

Whence, $a = 20$

3. If the ratio of y to x is always 5, what is the value of y when $x = 6$? What is the value of x when $y = 20$?

4. If y varies directly as x and $y = 35$ when $x = 20$, what is the ratio of the two numbers? What is the value of x when $y = 42$? What is the value of y when $x = 36$?

5. What is the ratio of the cost of 6 pounds of butter to the cost of 3 pounds of butter? What is the ratio of the cost of 5 oranges to the cost of a dozen oranges?

6. If 3 pounds of butter cost 75 cents, what will 6 pounds cost? 9 pounds? $4\frac{1}{2}$ pounds? 4 pounds? $7\frac{1}{2}$ pounds?

7. The ratio of the area of a circle to the square of its radius is about $\frac{22}{7}$ (called π). What is the area of a circle whose radius is 10 inches? 6 feet? 8 inches? 20 centimeters?

36.56 6 - 4 cm

8. The area of a rectangle whose base is of constant length varies directly as the altitude. For example, if A is the area, h the altitude, and the base is 5, we have

$$\frac{A}{h} = 5$$

Find A when h is 12; when h is 24; when h is 7.4.

Two numbers are said to vary **inversely** when their product is always the same. For example, if $xy = 12$ for all values of x, then x varies inversely as y and y varies inversely as x.

If x and y vary inversely and $x = 4$ when $y = 6$, then the relation between x and y is expressed by the equation $xy = 24$.

If $x = 2$, we have $2y = 24$. Whence $y = 12$.

EXERCISES

1. Two numbers vary inversely and one equals 6 when the other equals 3. What is the value of the second when the first equals 4?

Solution:

Let $x =$ the second number when the first is 4. Since the two numbers vary inversely, their product is 6 times 3, or 18. Then the product of 4 and x must be 18. That

is, $$4x = 18$$
and $$x = 4\tfrac{1}{2}$$

2. Two numbers vary inversely and one is 12 when the other is 15. If x and y represent the numbers, what equation represents the relation between them?

3. In problem 2, what is x when $y = 20$? when $y = 5$? when $y = 6$?

4. In formula (b), section 3, $c = np$, suppose the total cost is 50 cents. What does the formula become? How do the number of things bought (n) and the price of each (p) vary if the cost (c) is constant?

5. In the formula $A = bh$, if the area (A) is constant, how do the base (b) and the altitude (h) of the rectangle vary?

6. The dimensions of a rectangle change in such a way that the area is always equal to 80 square inches. What is the base when the altitude is 10 inches? What is the altitude when the base is 5 inches?

166. Proportion.

An equation whose members are ratios is called a **proportion.** Thus, $\frac{a}{b} = \frac{3}{5}$ is a proportion, and is sometimes written,

$$a : b = 3 : 5$$

This is read, the ratio of a to b equals the ratio of 3 to 5.

The first and last terms, a and 5, are called the **extremes,** and the middle terms, b and 3, the **means** of the proportion.

In the proportion $\frac{a}{b} = \frac{x}{y}$, what are the means and what are the extremes?

In $\frac{a}{b} = \frac{c}{d}$, d is said to be a **fourth proportional** to a, b, and c.

If $\frac{a}{b} = \frac{b}{c}$, b is said to be a **mean proportional** between a and c.

The following is a very important principle, or theorem, of proportion:

If four numbers are in proportion, the product of the means equals the product of the extremes.

Suppose $\frac{a}{b} = \frac{c}{d}$.

Multiplying both members by bd, $\frac{a}{b} \cdot bd = \frac{c}{d} \cdot bd$

Whence, $ad = bc$

If three terms of a proportion are known, the remaining term can be found.

If $\frac{x}{5} = \frac{6}{10}$

then by the above principle, $10x = 30$

Whence $x = 3$

If $\frac{300}{x} = \frac{2}{3}$

then $2x = 3 \cdot 300 = 900$

Whence, $x = 450$

EXERCISES

Find the unknown term in each of the following:

1. $\frac{a}{3} = \frac{8}{6}$

2. $\frac{4}{5} = \frac{y}{10}$

3. $\frac{2}{7} = \frac{6}{x}$

4. $\frac{s}{8} = \frac{3}{4}$

5. $\frac{5}{x} = \frac{2}{7}$

6. $\frac{12}{5} = \frac{s}{15}$

7. $\frac{16}{a} = \frac{4}{7}$

8. $\frac{3}{k} = \frac{6}{15}$

9. $\frac{7.5}{4} = \frac{m}{12}$

10. $\frac{x}{24} = \frac{3}{8}$

11. $\frac{3}{8} = \frac{15}{y}$

12. $\frac{18}{s} = \frac{16}{11}$

13. $\frac{2.25}{k} = \frac{5}{4}$

14. $\frac{3.6}{5} = \frac{m}{4}$

15. $\frac{14.4}{a} = \frac{3.6}{2.5}$

If $$am = bn \quad (1)$$

we may divide both members by bm and obtain

$$\frac{a}{b} = \frac{n}{m} \quad (2)$$

If we divide both members of (1) by mn, we have

$$\frac{a}{n} = \frac{b}{m} \quad (3)$$

We may also obtain

$$\frac{m}{b} = \frac{n}{a} \quad (4) \quad \text{(dividing both members by } ab\text{)}$$

and $$\frac{m}{n} = \frac{b}{a} \quad (5) \quad \text{(dividing both members by } an\text{)}$$

From the preceding we have the following principle:

If the product of two numbers equals the product of two other numbers, either pair of factors may be made the extremes of a proportion and the other pair of factors the means of the proportion.

In the proportions above, a and m are the extremes and b and n the means. If we interchange the members, that is, write $\frac{b}{m} = \frac{a}{n}$ instead of $\frac{a}{n} = \frac{b}{m}$, b and n are the extremes and a and m the means. It is evident that any one of the four factors may be taken as the first term of the proportion and when the first term is chosen, the last term is fixed, since the first and last terms must belong to the same product.

From the equation $4 \cdot 6 = 3 \cdot 8$, we have, by making 4 and 6 the extremes,

$$\frac{4}{3} = \frac{8}{6};\quad \frac{4}{8} = \frac{3}{6};\quad \frac{6}{3} = \frac{8}{4};\quad \frac{6}{8} = \frac{3}{4}$$

And by making 3 and 8 the extremes, we have

$$\frac{8}{6} = \frac{4}{3};\quad \frac{3}{6} = \frac{4}{8};\quad \frac{8}{4} = \frac{6}{3};\quad \frac{3}{4} = \frac{6}{8}$$

EXERCISES

Form two proportions from each of the following equations:

1. $9 \cdot 4 = 3 \cdot 12$
2. $5 \cdot 10 = 25 \cdot 2$
3. $6a = 8 \cdot 9$ $\frac{6}{9} = \frac{a}{8}$
4. $ax = mn$
5. $bc = 45$
6. $3xy = bm$ $\frac{3}{b} = \frac{m}{xy}$
7. $a^2 - b^2 = xy$
8. $s^2 - 4 = 24$
9. $a^2 - 8a - 9 = pq$
10. $x^3 - y^3 = 48$
11. $x^2 = 2 \cdot 8\ (x^2 = x \cdot x)$
12. $8^2 = ab$ $\frac{a}{8}\ \frac{8}{a}$
13. $x^2 = (a + b)(a - b)$
14. $m^2 = (a^2 - 25)$
15. $(a - x)^2 = 45$
16. Find a mean proportional between 2 and 8.

Solution:

Let x be the required mean proportional.

Then, $$\frac{2}{x} = \frac{x}{8}$$

Whence, $$x^2 = 16$$

and $$x = \sqrt{16} = 4$$

Find the mean proportionals between the following numbers:

17. 5 and 125
18. 4 and 64
19. 6 and 216
20. 1 and $\frac{1}{4}$
21. $\frac{2}{3}$ and $\frac{2}{27}$
22. a and a^3

Find the fourth proportional to:

23. 3, 5, and 6

24. 4, 7, and 12

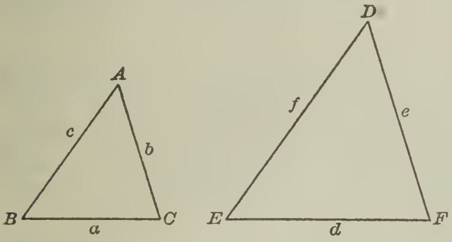
25. 27, 21, and 36

26. .6, 2.4, and 1.5

27. 7, 12, and 8

28. 5, 2.5, and 10

The corresponding sides of similar triangles are proportional. If ABC and DEF are similar triangles, then



$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

or, using small letters to represent the number of linear units in the sides,

$$\frac{c}{f} = \frac{a}{d} = \frac{b}{e}$$

29. In two similar triangles $c = 28$, $f = 20$, and $d = 30$. What is the length of a?

30. In two similar triangles, $a = 24$ inches, $d = 30$ inches, and $b = 48$ inches. What is the length of e?

31. Two corresponding sides of two similar triangles are 8 feet and 20 feet respectively. A second side of the first triangle is 18 feet. What is the length of the corresponding side of the second triangle?

32. The sides of a triangle are 60, 75, and 80. The longest side of a similar triangle is 120. (The longest sides, 80 and 120, are corresponding sides.) Find the lengths of the other sides.

33. The ratio of two corresponding sides of two similar triangles is $\frac{3}{5}$. One side of the smaller triangle is 45 inches. What is the length of the corresponding side of the larger triangle?

The circumferences of two circles have the same ratio as their radii, and the areas of two circles have the same ratio as the squares of their radii.

34. The radii of two circles are 8 inches and 12 inches respectively. What is the ratio of their circumferences? The circumference of the smaller circle is approximately 50.266 inches. What is the circumference of the larger circle?

35. The radii of two circles are 9 inches and 16 inches respectively. The sum of their circumferences is 157.08 inches. What is the circumference of each? (Let $9x$ represent the circumference of the smaller circle. What will represent the circumference of the larger circle?)

36. The circumferences of two circles are 37.68 rods and 62.8 rods respectively. If the radius of the larger circle is 10 rods, what is the radius of the smaller?

37. The radii of two circles are 8 inches and 15 inches respectively. What is the ratio of their areas?

38. The radii of two circles are 6 feet and 10 feet respectively. The area of the smaller circle is 113.04 square feet. What is the area of the larger?

39. The ratio of the areas of two similar triangles equals the ratio of the squares of two corresponding sides. Two corresponding sides are 3 and 5. What is the ratio of the areas of the two triangles? $9/25$

✓ **40.** Two corresponding sides of two similar triangles are 12 inches and 18 inches respectively. If the area of the first triangle is 72 square inches, what is the area of the second?

$$\frac{144}{324} = \frac{72}{x}$$

✓ **41.** The lengths of shadows are proportional to the heights of objects casting them. A certain tree casts a shadow 48 feet long when the shadow cast by a pole 9 feet high is 8 feet long. What is the height of the tree?

9| $\frac{9}{8} = \frac{x}{48}$

42. At the same time the shadows in problem 41 were measured, the shadow cast by the smoke stack of a school building was 56 feet in length. What is the height of the smokestack? $\frac{9}{8} = \frac{x}{56}$

✓ **43.** A 6-foot pole casts a shadow $4\frac{1}{2}$ feet in length. At the same time the shadow of a certain tree is 42 feet in length. What is the height of the tree?

✓ **44.** The shadow of a certain tree is 60 feet in length at the same time that the shadow of an 8-foot pole is 10 feet in length. What is the height of the tree?

✓ **45.** What is the height of the Washington Monument if it casts a shadow 333 feet long at the same time that a pole 10 feet high casts a shadow 6 feet long?

SUMMARY

SECTION
164. Ratio. Problems.
165. Variation. Definitions. Problems.
166. Proportion. Definitions. Problems.

56
9
504

42
6
252

$\frac{16}{9}$ 56 72

4.5)252

23328

INDEX

Page
- Absolute Value.............. 35
- Addition.............20, 33, 48
  - Commutative Law of..... 51
  - of Fractions.............170
  - of Polynomials........... 54
  - of Radicals..............318
- Ahmes..................... 15
- Algebraic Expression, defined. 9
  - Interpretation of......... 9
  - Value of................. 10
- Antecedent.................333
- Applications of Square Root..264
- Arithmetic Mean............ 44
- Associative Law of Multiplication................ 85
- Averages................... 43
- Axioms..................... 59

- Binomial, defined........... 48
  - Square of................125
  - Cube of..................250

- Coefficient.................. 8
  - Literal..................197
- Commutative Law of Addition.................. 51
  - of Multiplication......... 85
- Complete Divisor............255
- Completing the Square ..128, 276
- Complex Fractions...........189
  - Number...................287
- Conditional Equations and Identities............. 78
- Consequent.................333
- Cube of Binomial............250

- Degree of Term.............106
- Denominator, Rationalizing the..................325
- Determinate Solution........223
  - System of Equations......223
- Division..............25, 42, 95
  - of Fractions..........180, 187
  - of Radicals..............323

- Elimination, defined.........223
  - by Addition and Subtraction..................222
  - by Substitution..........226

Page
- Ellipse....................301
- Equations..............60, 106
  - Conditional and Identities 78
  - Containing Fractions 194
  - Containing Radicals......328
  - Incompatible or Inconsistent...............245
  - Linear...................107
  - Quadratic................273
  - Simple...................107
  - Simultaneous........220, 223
  - Solution of............60, 78
  - Solution of by Factoring..148
- Evaluation................. 11
- Evolution..................251
- Exponent................8, 305
  - Fractional...............308
  - Negative.................306
  - Positive Integral.........305
  - Zero.....................306

- Extension of Index Laws.....306
- Factor, Common............155
  - Highest Common.........155
  - Rationalizing............325
- Factors.................. 7, 113
- Formula, defined............ 5
  - Substitution in........... 12
- Fractions...................162
  - Complex.................189
  - Equations Containing....194

- General Numbers............ 4
- Graph of an Equation........212
- Graphs.....................205

- Highest Common Factor.....155
- Hyperbola..................302

- Identities and Conditional Equations............ 78
- Imaginary Numbers.........287
- Incompatible or Inconsistent Equations...........245
- Index.....................247
  - Law of Evolution.........252
  - Law of Involution........253
- Integral Terms..............106
- Involution..................247

Page
Law, Associative, of Multiplication................. 85
  Commutative, of Addition 51
  of Multiplication......... 85
  of Exponents in Division.. 95
Literal Coefficients...........197
  Numbers...............4, 18
Lowest Common Multiple....158

Mean, Arithmetic........... 44
Members of Equations....... 60
Monomial...............48, 85
Multiple, Lowest Common ... 58
Multiplication.........24, 41, 85
  of Fractions..........178, 183
  of Radicals..............320

Negative Exponent.........306
  Numbers................. 31
Numbers, Complex..........287
  General.................. 4
  Imaginary...............287
  Literal.................4, 18
  Real....................287
Operation, Symbols of........ 7
Operations, Signs of the Fundamental............. 26
Order and Grouping, Laws of. 51
Order, Law of, in Multiplication................. 85

Parabola...................300
Parentheses, Insertion of..... 76
  Removal of.............. 73
Polynomial................. 48
Polynomials, Addition of..... 54
  Division of, by Monomial. 97
  by Polynomial........... 98
  Factored by grouping Terms................115
  Multiplication of, by Monomials................. 88
  by Polynomials........... 89
  Subtraction of........... 69

Positive and Negative Numbers, Addition of...... 33
  Division of.............. 42
  Multiplication of......... 41
Positive Integral Exponent...305
Power...................... 9
  of a Monomial...........247

Page
Proportion.............333, 337

Quadratic Equations.........273
  Affected.................273
  Complete................273
  Graphs of...............299
  Incomplete..............273
  Pure....................273
  Simultaneous............296

Radicals...............305, 314
  Addition and Subtraction of...................318
  Division of.............323
  Equations Containing.....328
  Multiplication of.........320
Ratio.....................333
Rationalizing Factor.........325
  the Denominator.........325
Reciprocal of a Number......187
Reduction of Fractions to Lowest Terms.........164
  to Common Denominator .166
Removal of Parentheses 73
Representation, Graphical, of Distance.............205
  of Other Number Relations 206
Signs of the Fundamental Operations............ 26
Similar or Like Terms........ 55
Simple Equations............107
Simultaneous Equations......220
  Quadratics...............296
Solution, Determinate........223
  of Equations...........60, 78
  by Factoring.............148
  of Quadratic Equations
  by Completing the Square 276
  by Factoring.............273
  by Formula..............282
Special Products and Quotients................113
Square, Completing the Square...........128, 276
  of Binomial.............125
  Polynomial..............248
Square Root, Applications of.264
  of Arithmetical Numbers .258
  of Fractions.............262
  of Polynomials...........254

Page
Squares, Difference of, Factors of....................131
  Trinomial, Factors of.....126
Substitution, Elimination by.226
  in Formula.............. 12
Subtraction................. 23
  of Positive and Negative Numbers............. 37
Subtraction of Monomials.... 67
Sum of Two Cubes..........137
  Factors of...............139
Symbols of Grouping....... 9, 73
  of Operation............. 7
Temperature and Other Graphs..............208

Page
Term...................... 9
  Degree of................106
Terms, Integral.............106
  Similar or Like........... 55
Trinomial.................. 48
  Squares, Factors of.......126

Value, Absolute............. 35
  of Algebraic Expression... 10
Variables...................218
Variation...................334

Zero Exponent..............306

x = wheat

$2x$ = Oats

$8x$ = Corn

$11x = 1683$

$x = 153$ wheat

$2x = 306$ Oats

$8x = 1224$ Corn

1683

x = wt of [illegible] Quarry marble

$x + 47$ " " " glass

$x + 53$ " " "

$3x - 15 = x + 47 + x + 53 + 15$

$3x - x - x = 53 + 47 + 15$

$x = 115$ D

$x + 47 = 162$ m

$x + 53 = 168$ G.

Trinomials
grouping
Perfect Square
Trinomials
Difference of [illegible]
Trial [illegible]
Sum of

1^3 1
2^3 8
3^3 27
4^3 64
5^3 125
6^3 216
7^3 343
8 512
9^3 729
10^3 1000

+ 24
− 18
+ 6

3
12
144

288
864
1728
2266

14
18
112
34
432